3 STRAIGHT LINES (ANGLE BETWEEN LINES, FOOT, IMAGE, LOCUS)

SYNOPSIS

1. If 'θ' is an acute angle between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ then

$\cos\theta=\left|\dfrac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}\right|$, $\sin\theta=\left|\dfrac{a_1b_2-a_2b_1}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}\right|$ and $\tan\theta=\left|\dfrac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right|$ other angle between the lines is $\pi-\theta$

2. If 'θ' is an acute angle between the lines having slopes m_1 *and* m_2 then $tan\theta=\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

Note : The slopes of the lines making an angle α with a line having slope m are

$\dfrac{m-\tan\alpha}{1+m\tan\alpha},\dfrac{m+\tan\alpha}{1-m\tan\alpha}$

3. Consider two lines $L_1=a_1x+b_1y+c_1=0$ and $L_2=a_2x+b_2y+c_2=0$

i) Lines are parallel if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$

ii) Lines are coincident if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

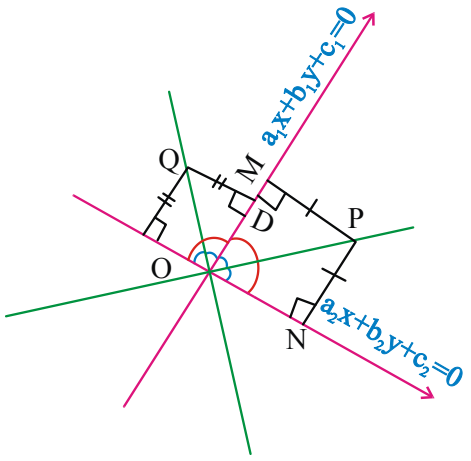
iii) Lines are perpendicular if $a_1a_2+b_1b_2=0$

iv) Lines are equally inclined with x-axis *if* $\dfrac{a_1}{a_2}=-\dfrac{b_1}{b_2}$

4. **Angular bisectors of two straight lines:**

Angular bisector is the locus of a point which moves in such a way that its distance from two intersecting lines remains same.

i) The equations of the two bisectors of the angles between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$



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ii) If the two given lines are not perpendicular i.e. $a_1a_2 + b_1b_2 \neq 0$ and not parallel i.e. $a_1b_2 \neq a_2b_1$ then one of angular bisectors is the bisector of the acute angle between two given lines and the other that of the obtuse angle between two given lines.

iii) Whaether both given lines are perpendicular or not, but the angular bisectors of given lines will always be mutually perpendicular.

5. The bisector of the angle which contains a given point :

$L_1 = a_1x + b_1y + c_1 = 0$, $L_2 = a_2x + b_2y + c_2 = 0$ are two intersecting lines then equations of lines bisecting angles between $L_1 = 0, L_2 = 0$ are

$$\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right) = \pm\left(\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right) \rightarrow (1)$$

and $P(h,k)$ be a point

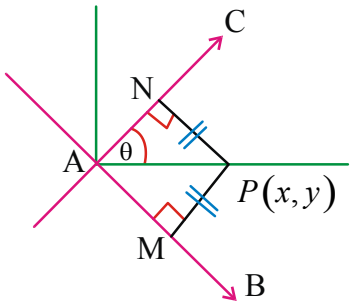
i) If $L_1(h,k), L_2(h,k)$ are having same sign then (+) sign in RHS of (1) gives bisector of angle containing the point $P(h,k)$ (–) sign in RHS of (1) gives bisector of angle not containg the point $P(h,k)$

ii) If $L_1(h,k), L_2(h,k)$ are opposite in sign then (–) sign in RHS of (1) gives bisector of angle containing the point $P(h,k)$ (+) sign in RHS of (1) gives bisector of angle not containing the point $P(h,k)$

iii) If C_1, C_2 have same sign then (+) sign in RHS gives bisector of angle containing origin (–) sing in RHS gives bisector of angle not containing origin

6. The bisectors of the acute and the obtuse angles between two lines :

Take one of the two lines and let its slope be m_1 and take one of the angle bisectors and let its slope be m_2. If θ be the acute angle between them, then find $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$



If $\tan\theta > 1$ then the bisector taken is the bisector of the obtuse angle and the other one will be the bisector of the acute angle.

If $0 < \tan\theta < 1$ then the bisector taken is the bisector of the acute angle and the other one will be the bisector of the obtuse angles.

(or)

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$L_1 = a_1x + b_1y + c_1 = 0$, $L_2 = a_2x + b_2y + c_2 = 0$ are two intersecting lines where C_1, C_2 have same sign

Condition	Obtuse angle bisector	Acute angle bisector
$a_1a_2 + b_1b_2 > 0$	$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{+(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$ Bisector of angle containg origin	$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\left(\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right)$ Bisector of angle not containg origin
$a_1a_2 + b_1b_2 < 0$	$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{-(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$ Bisector of angle not containg origin	$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\left(\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right)$ Bisector of angle containg origin

7. Equation of straight lines passing through $P(x_1, y_1)$ and equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines and passing through the point P.

8. Nature of triangles and Quadrilateral :

i) The ratio of the sides of a triangle formed by $L_1 = 0,\ L_2 = 0\ and\ L_3 = 0$ is

$\sqrt{a_1^2 + b_1^2}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} : \sqrt{a_2^2 + b_2^2}\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} : \sqrt{a_3^2 + b_3^2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ From this use side nature of the Δ^{le}

ii) Let d_1 be the distance between the parallel lines

$ax + by + c_1 = 0$ ------(1)

$ax + by + c_2 = 0$ -------(2)

and d_2 be the distance between the parallel lines $a_1x + b_1y + k_1 = 0$ ------(3)

$a_1x + b_1y + k_2 = 0$ ----------(4)

then the figure formed by four lines is

a) a square if $d_1 = d_2$ and $aa_1 + bb_1 = 0$,

b) Rhombus if $d_1 = d_2$ and $aa_1 + bb_1 \neq 0$,

c) Rectangle if $d_1 \neq d_2$ and $aa_1 + bb_1 = 0$,

d) Parallelogram if $d_1 \neq d_2$ and $aa_1 + bb_1 \neq 0$

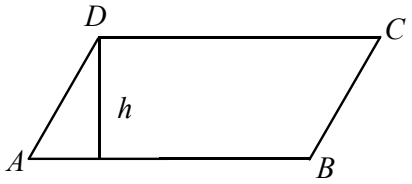
9. **i)** The area of triangle formed by lines $a_ix + b_iy + c_i = 0,\ i = 1,2,3$ is $= \frac{\Delta^2}{2|\lambda_1\lambda_2\lambda_3|}$

$$where\ \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \lambda_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},\ \lambda_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \lambda_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

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ii) The area of triangle formed by lines $y = m_i x + c_i,\ i = 1,2,3$ *is* $\frac{1}{2}\left|\sum \frac{(c_1 - c_2)^2}{m_1 - m_2}\right|$

10. **i)** Area of the parallelograum ABCD $= (AB) \times h =$ Base $\times$ height



ii) If p_1, p_2 are distances between parallel sides and 'θ' is angle between adjacent sides of parallelogram then its area is $\left|\frac{p_1 p_2}{\sin\theta}\right|$

iii) Area of parallelogram whose sides are $a_1x + b_1y + c_1 = 0,\ a_1x + b_1y + c_2 = 0,$ $a_2x + b_2y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ is $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{a_1b_2 - a_2b_1}\right|$

iv) Area of rhombus $= \frac{1}{2}\ d_1d_2$ where d_1, d_2 are lengths of the diagonals

v) Area of the rhombus $a|x| + b|y| + c = 0$ is

11. Foot of the perpendicular from the point (x_1, y_1) **on**

i) a line $ax + by + c = 0$ is (h, k) then $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

ii) X – axis is $(x_1, 0)$

iii) Y – axis is $(0, y_1)$

iv) x = k is (k, y_1)

v) y = k is (x_1, k)

12. Image of the point A w.r.t. the point P is 2P–A

13. Image of a point (x_1, y_1) w.r.t.

i) a line $ax + by + c = 0$ is (h, k) then $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$

ii) X – axis is $(x_1, -y_1)$

iii) Y – axis is $(-x_1, y_1)$

iv) x = k is $(x_1, 2k - y_1)$

v) y = k is $(2k - x_1, y_1)$

vi) y = x is (y_1, x_1)

vii) y = – x is $(-y_1, -x_1)$

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14. **i)** Reflection of $f(x,y)=0$ in x-axis is $f(x,-y)=0$

ii) Reflection of $f(x,y)=0$ in y-axis is $f(-x,y)=0$

iii) Reflection of $f(x,y)=0$ in x = y is $f(y,x)=0$

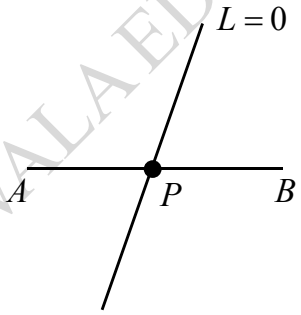
iv) Image of the line ax+by+c = 0 w.r.t line lx + my + n =0 is (or) the straight line lx + my + n = 0 bisects an angle between the two lines of which one of them is ax + by + c = 0 then equation of other line is $\frac{ax+by+c}{lx+my+n}=\frac{2(al+bm)}{l^2+m^2}$

15. **i)** A straight line passing through the point (x_1,y_1) meets the positive coordinate axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$ at A and B then the minimum value of $OA+OB=\left(\sqrt{x_1}+\sqrt{y_1}\right)^2$

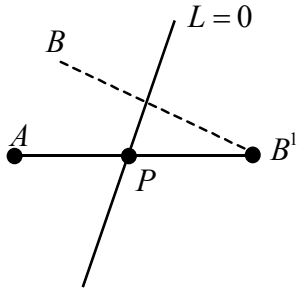
ii) A straight line passing through (x_1,y_1) meets the co – ordinate axes at A, B then the area of the Δ^{le} OAB is minimum if (x_1,y_1) is the midpoint of $\overline{AB}$ and that minimum area is $\frac{1}{2}|x_1,y_1|$

16. If A, B are two fixed points and P be a point on L = 0.

i) A, B lie on opposite side of L = 0 and PA + PB is minimum then P, A, B are collinear and P is the P.O.I. of $\overline{AB}$ and L = 0 and minimum value of PA + PB = AB

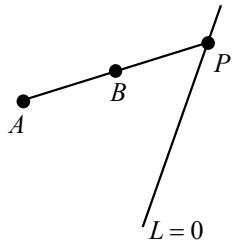


ii) A, B lie on same side of L = 0 and PA + PB is minimum then P, A, B^1 are collinear, where B^1 is the image point of B w.r.t the line L = 0 and P is P.O.I. of $\overline{AB}^1$ and L = 0 and minimum value of $PA+BP=AB^1$

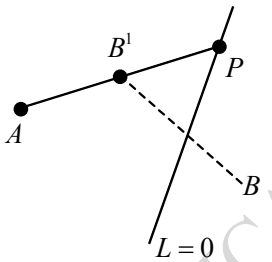


iii) A, B lie on the same side of L = 0 and $|PA-PB|$ is maximum then P, A, B are collinear and P is the P.O.I. of $\overline{AB}$ and L = 0. The maximum value of $|PA-PB|=AB$

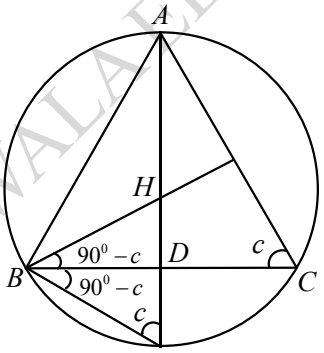
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iv) A, B lie on the opposite side of L = 0 and $|PA - PB|$ is maximum then P, A, B^1 are collinear where B^1 is the image point of B w.r.t the line L = 0 and P = P.O.I. of the $\overline{AB}^1$ and L = 0 and the maximum value of $|PA - PB| = AB^1$



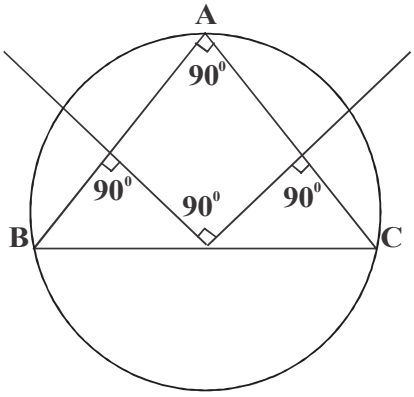
17. Image of orthocentre of ΔABC w.r.t. a side of the triangle lies on circumcircle of ΔABC



18. If the perpendicular bisectors of sides $\overline{AB}$, $\overline{AC}$ are perpendicular then

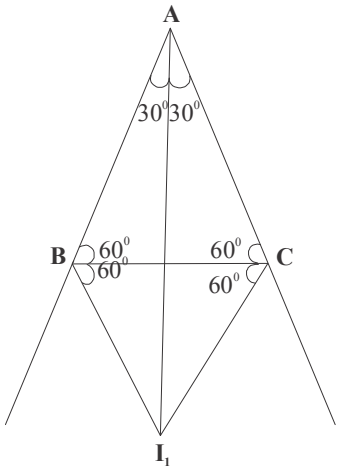
i) $\angle BAC$ **is** 90^0

ii) point of intersection of the perpendicular bisectors is mid point of $\overline{BC}$



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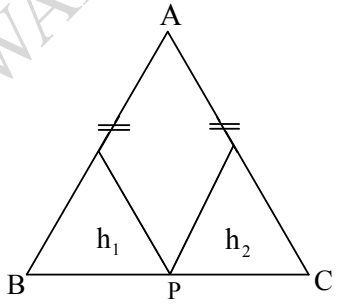
19. For an equilateral ΔABC, ex-centres I_1, I_2 and I_3 are images of A,B and C w.r.t. $\overline{BC}, \overline{CA}$ & $\overline{AB}$ respectively



20. In a right angled triangle, the sum of the lengths of the legs is equal to the sum of the diameters of the inscribed and the circumscribed circles.

21. If the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of a triangle ABC are divided by the points D,E,F in the same ratio, then the centroids of $\Delta ABC, \Delta DEF$ are coincide.

22. In an isosceles triangle the sum of the distances from any point of the base to the lateral sides is constant.



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23. **Locus :** Locus is the set of points (and only those points) that satisfy the given consistant geometric condition(s).
i.e **i)** Every point satisfying the given condition (s) is a point on the locus.
ii) Every point on the locus satisfies the given condition(s).

24. **Working Procedure to find equation to a Locus:-**

Step-1:- Let $P(x_1, y_1)$ be a point in a Locus

Step-2 :- Write the given geomentrical condition

Step-3:- Convert the geomentrial condition into Algebraic Equation

Step - 4:- Replace (x_1, y_1) by (x, y) in above algebraic equation, the equation so obtained is called equation to the Locus

25. **i)** The locus of a point which is equidistant from two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the perpendicular bisector of the line segment AB whose equation is

$$(x_1 - x_2)x + (y_1 - y_2)y = (x_1^2 - x_2^2) + (y_1^2 - y_2^2)$$

ii) Locus of P which is collinear with A, B is the line $\overleftrightarrow{AB}$.

26. **i)** The locus of a point which is at a distance 'r' units from the given point $A(x_1, y_1)$ is a circle whose equation is $(x-x_1)^2+(y-y_1)^2=r^2$

ii) If the join of two fixed points $(x_1,y_1),(x_2,y_2)$ subtends a right angle at P, then the locus of P is a circle on AB as diameter, whose equation is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

27. A and B are fixed points. P is the point moves such that $\frac{PA}{PB}=k$ is

i) a straight line if k=1

ii) a circle if $k \neq 1$ and k>0.

iii) an empty set if $k<0$

28. If A, B,C are three points then the locus of a point P such that $PA^2+PB^2=K.PC^2$ is

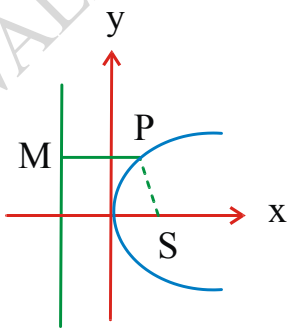
i) a straight line if K=2

ii) a circle if $k \neq 2$ and K>0

iii) an empty set if $k<0$

29. Given A & B are two fixed points.The locus of a point P such that the area of Δ PAB is a constant is a pair of lines parallel to AB and the distance between the parallel lines is $\frac{4\Delta}{AB}$, Δ is the area of ΔPAB

30. The locus of the point which moves equidistant from a fixed point and fixed st. line is a parabola.



31. A,B are two fixed points and
PA + PB = k then

i) If AB < k, locus of P is an ellipse

ii) If AB =k, locus of P is line segment AB

iii) If AB >k , locus of P does not exist

32. A,B are two fixed points and
|PA - PB| = k, then

i) If AB <k, locus of P does not exist

ii) If AB =k, locus of P is line through A and B except line segment AB

iii) If AB > k, locus of P is a hyperbola

33. The curve represented by $S=ax^2+2hxy+by^2+2gx+2fy+c=0$

i) a pair of lines if $\Delta=0$, $h^2 \geq ab, g^2 \geq ac$, $f^2 \geq bc$

ii) a pair of parallel lines if $\Delta=0$, $h^2=ab$, $af^2=bg^2$

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iii) a circle if a=b, h=0, $g^2 + f^2 - ac \geq 0$, $\Delta \neq 0$

iv) a parabola if $\Delta \neq 0$, $h^2 = ab$.

v) an ellipse if $\Delta \neq 0, h^2 < ab$.

vi) a hyperbola if $\Delta \neq 0, h^2 > ab$

vii) a rectangular hyperbola if $\Delta \neq 0$, a+b=0 and $h^2 > ab$ where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

34. The locus of a point

i) $(x_1 + r\cos\theta, y_1 + r\sin\theta)$, r is a parameter is a straight line, whose equation is

$y - y_1 = \tan\theta(x - x_1)$

ii) $(x_1 + r\cos\theta, y_1 + r\sin\theta), \theta$ is a parameter is a circle, whose equation is

$(x - x_1)^2 + (y - y_1)^2 = r^2$

iii) $(at^2, 2at), t$ is a parameter is a parabola, whose equation is $y^2 = 4ax$

iv) $(a\cos\theta, b\sin\theta), \theta$ is a parameter is an ellipse, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

v) $(a\cosh\theta, a\sinh\theta), \theta$ is a parameter is a hyperbola, whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

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