13 MULTIPLES AND SUBMULTIPLE ANGLES

SYNOPSIS

1. Let A be an angle then the multiple of A are 2A, 3A, 4A, ... etc and the sub multiples of A are $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$..... etc.

2. **i)** $\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$

ii) $\cos 2A = \cos^2 A - \sin^2 A = 1-2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$

iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

3. **i)** $2\sin^2 A = 1-\cos 2A, 2\cos^2 A = 1+\cos 2A$

ii) $\sin^2 A = \frac{1-\cos 2A}{2}$, $\cos^2 A = \frac{1+\cos 2A}{2}$, $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

iii) $\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}$, $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$, $\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$

4. $\sin 3A = 3\sin A - 4\sin^3 A$
$\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

5. **i)** $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1-2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

iii) $\tan A = \frac{2\tan^2\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

6. **i)** $2\sin^2\frac{A}{2} = 1-\cos A, 2\cos^2\frac{A}{2} = 1+\cos A$

ii) $\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$, $\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$, $\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A}$

iii) $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

7. $\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$, $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$, $\tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$

8. **i)** $\sin 18^0 = \frac{\sqrt{5}-1}{4} = \cos 72^o$ **ii)** $\cos 18^0 = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^o$

iii) $\sin 36^o = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^o$ **iv)** $\cos 36^o = \frac{\sqrt{5}+1}{4} = \sin 54^o$

9. **i)** $\sin 22\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = \cos 67\frac{1}{2}^0$ **ii)** $\cos 22\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sin 67\frac{1}{2}^0$

iii) $\tan 22\frac{1}{2}^0 = \sqrt{2}-1 = \cot 67\frac{1}{2}^0$ **iv)** $\cot 22\frac{1}{2}^0 = \sqrt{2}+1 = \tan 67\frac{1}{2}^0$

10. **i)** $\tan 7\frac{1}{2}^0 = \sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2} = \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right)$

ii) $\cot 7\frac{1}{2}^0 = \sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2} = \left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$

iii) $\tan 37\frac{1}{2}^0 = \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}+1\right) = \cot 52\frac{1}{2}^0$

iv) $\cot 37\frac{1}{2}^0 = \left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}-1\right) = \tan 52\frac{1}{2}^0$

11. **i)** $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta = 1-\sin^2\theta\cos^2\theta$

ii) $\sin^4\theta + \cos^4\theta = 1-2\sin^2\theta\cos^2\theta = 1-\frac{1}{2}\sin^2 2\theta$

iii) $\sin^6\theta + \cos^6\theta = 1-3\sin^2\theta\cos^2\theta = 1-\frac{3}{4}\sin^2 2\theta$

12. **i)** $\tan\left(\frac{\pi}{4}+A\right) + \tan\left(\frac{\pi}{4}-A\right) = 2\sec 2A$

ii) $\tan\left(\frac{\pi}{4}+A\right) - tan\left(\frac{\pi}{4}-A\right) = 2\tan 2A$

iii) $\tan\left(\frac{\pi}{4}+A\right).Tan\left(\frac{\pi}{4}-A\right) = 1$

13. **i)** cotA - tan A = 2 cot 2A
ii) cotA + tanA = 2 cosec 2A
iii) cotA = cosec 2A + cot 2A
iv) tan A = cosec 2A - cot 2A

v) $\tan A + 2\tan 2A + ...+ 2^{n-1}\tan 2^{n-1}A + 2^n \cot 2^n A = \cot A$

vi) $\cot A - \tan A - 2\tan 2A.....-2^{n-1}\tan 2^{n-1}A = 2^n \cot 2^n A \left(A \neq \frac{n\pi}{2}, n \in Z\right)$

14. i) $\tan\frac{\theta}{2}(1+\sec\theta) = \tan\theta$

ii) $\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta) = \tan 2\theta$

iii) $\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta).....(1+\sec 2^n\theta) = \tan 2^n\theta$

15. i) $(2\cos\theta - 1)(2\cos\theta + 1) = 2\cos 2\theta + 1$

ii) $(2\cos\theta - 1)(2\cos\theta + 1)(2\cos 2\theta - 1) = 2\cos 4\theta + 1$

iii) $(2\cos\theta - 1)(2\cos 2\theta - 1) = \frac{2\cos 4\theta + 1}{2\cos\theta + 1}$

iv) $(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 4\theta - 1).....(2\cos 2^{n-1}\theta - 1) = \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1}$

16. i) $\cos\theta.\cos 2\theta.\cos 4\theta = \frac{\sin 8\theta}{8\sin\theta}$

ii) $\cos\theta.\cos 2\theta.\cos 4\theta.\cos 8\theta = \frac{\sin 16\theta}{16\sin\theta}$

iii) $\cos\theta.\cos 2\theta.\cos 4\theta.\cos 8\theta.\cos 16\theta = \frac{\sin 32\theta}{32\sin\theta}$

iv) $\cos\theta.\cos 2\theta.\cos 2^2\theta.....\cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}$

17. If $a_1, a_2, a_3, ... a_n$ are in AP with common difference 'd' then

i) $\cos a_1 + \cos a_2 + \cos a_3 + ... + \cos a_n = \frac{\cos\left(\frac{a_1 + a_n}{2}\right)\sin\left(\frac{nd}{2}\right)}{\sin\frac{d}{2}}$

ii) $\sin a_1 + \sin a_2 + \sin a_3 + ... + \sin a_n = \frac{\sin\left(\frac{a_1 + a_n}{2}\right)\sin\left(\frac{nd}{2}\right)}{\sin\frac{d}{2}}$

18. i) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \frac{-1}{2}$

ii) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{6\pi}{7}\cos\frac{2\pi}{7} = \frac{-1}{2}$

iii) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} = \frac{1}{8}$

iv) $\text{cosec}^2\frac{\pi}{7} + \text{cosec}^2\frac{2\pi}{7} + \text{cosec}^2\frac{3\pi}{7} = 8$

v) $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}=\frac{\sqrt{7}}{8}$

vi) $\tan\frac{\pi}{7}+\tan\frac{2\pi}{7}+\tan\frac{4\pi}{7}=\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}=-\sqrt{7}$

vii) $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}+\tan\frac{3\pi}{7}\tan\frac{4\pi}{7}+\tan\frac{4\pi}{7}\tan\frac{\pi}{7}=-7$

viii) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}=\frac{1}{64}$

ix) $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{-1}{4}$

19. If x = 60^0 (or) 120^0 (or) 240^0 (or) 300^0 then

i) $\sin\theta\ \sin(x+\theta)\sin(x-\theta)=\frac{1}{4}\sin 3\theta$

ii) $\cos\theta\ \cos(x+\theta)\cos(x-\theta)=\frac{1}{4}\cos 3\theta$

iii) $\tan\theta\ \tan(x+\theta)\tan(x-\theta)=tan3\theta$

iv) $\cot\theta\ \cot(x+\theta)\cot(x-\theta)=\cot 3\theta$

20. If x = 60^0 (or) 120^0 (or) 240^0 (or) 300^0 then

i) $\tan\theta+\tan(x+\theta)-\tan(x-\theta)=3\tan 3\theta\ =3\tan\theta\ \tan(x+\theta)\tan(x-\theta)$

ii) $\cot\theta+\cot(x+\theta)-\cot(x-\theta)=3\cot 3\theta\ =3\cot\theta\ \cot(x+\theta)\cot(x-\theta)$

iii) $\cot(x+\theta)\cot(x-\theta)+\cot(x-\theta)\cot\theta\ -\cot(x+\theta)\cot\theta=3$

i.e $\cot\theta\ \cot(x-\theta)+\cot(x-\theta)\cot(x+\theta)-\cot(x+\theta)\cot\theta=3$

iv) $\tan\theta\ \tan(x-\theta)+\tan(x-\theta)\tan(x+\theta)-\tan(x+\theta)\tan\theta=3$

21. **i)** $\cos\theta+\cos(120^0+\theta)+\cos(120^0-\theta)=0$

ii) $\cos^3\theta+\cos^3(120^0+\theta)+\cos^3(120^0-\theta)=\frac{3}{4}\ \cos 3\theta$

iii) $\sin\theta+\sin(120^0+\theta)-\sin(120^0-\theta)=0$

iv) $\sin^3\theta+\sin^3(120^0+\theta)-\sin^3(120^0-\theta)\ =\frac{-3}{4}\sin 3\theta$

22. **i)** $\sin^2\theta+\sin^2(60^0+\theta)+\sin^2(60^0-\theta)=\frac{3}{2}$

ii) $\cos^2\theta+\cos^2(60^0+\theta)+\cos^2(60^0-\theta)=\frac{3}{2}$

In place of 60^0 we replace 120^0 (or) 240^0 (or) 360^0

23. If $A+B=60^o$ then

i) $\sin^2 A+\sin^2 B+\sin A\sin B=\frac{3}{4}$

ii) $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$

24. If $A-B=60^\circ$ then

i) $\sin^2 A+\sin^2 B-\sin A\sin B=\frac{3}{4}$

ii) $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$

25. **i)** $\frac{x}{y}=\frac{\sin A}{\sin B}$ then $\frac{x+y}{x-y}=\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$

ii) $\frac{x}{y}=\frac{\tan A}{\tan B}$ then $\frac{x+y}{x-y}=\frac{tanA+\tan B}{\tan A-\tan B}=\frac{\sin(A+B)}{\tan(A-B)}$

26. **i)** $\sqrt{1+\sin 2\mathrm{A}}=\left|\cos \mathrm{A}+\sin \mathrm{A}\right|$ **ii)** $\sqrt{1+\sin \mathrm{A}}=\left|\cos\frac{\mathrm{A}}{2}+\sin\frac{\mathrm{A}}{2}\right|$

iii) $\sqrt{1-\sin 2\mathrm{A}}=\left|\cos \mathrm{A}-\sin \mathrm{A}\right|$ **iv)** $\sqrt{1-\sin \mathrm{A}}=\left|\cos\frac{\mathrm{A}}{2}-\sin\frac{\mathrm{A}}{2}\right|$

27. Let $C=\cos\frac{A}{2};\ S=\sin\frac{A}{2}$, then

i) $c+s>0\ in\left(-\frac{\pi}{4},\frac{3\pi}{4}\right)$

ii) $c+s<0\ in\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$

iii) $c-s>0\ in\left(-\frac{3\pi}{4},\frac{\pi}{4}\right)$

iv) $c-s<0\ in\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

