# 6 LINEAR EQUATIONS

***SYNOPSIS***

**1.** **Sub matrix:** A matrix obtained by deleting some rows or columns (or both) of a matrix is called a submatrix.

***Ex.*** If $A=\begin{bmatrix}1 & 4 & 3\\ 6 & 3 & 1\\ -1 & 7 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\ 6 & 1\end{bmatrix}$ then $B$ is submatrix of $A$ ($B$ is also called two rowed minor of $A$).

**Note : i)** Every element of a matrix is a submatrix of order 1.

**ii)** Every matrix is a submatrix of itself.

**2.** **Rank of a matrix :** Let A be a non-zero matrix. The rank of $A$ is defined as the maximum of the orders of the non-singular square submatrices of $A$. The rank of a null matrix is defined as zero. The rank of a matrix $A$ is denoted by rank ($A$).

***Note :*** $A$ is a non-zero matrix of order 3, then the rank of $A$ is

**i)** 3 if $A$ is non-singular.

**ii)** 2 if $A$ is singular and there exist atleast one of its $2 \times 2$ submatrices is non-singular.

**iii)** 1 if $A$ is singular and every $2 \times 2$ submatrix is singular.

**3.** **Echelon form :** A matrix $A$ is said to be in Echelon form if the number of zeros before the first nonzero element in a row is less than the number of such zeros in the next row and every row of A with has all of its elements zero occurs below every row with has nonzero element.

***Ex.*** If $A=\begin{bmatrix}2 & 3 & 1 & 5 & 7\\ 0 & 3 & 5 & 6 & 2\\ 0 & 0 & 0 & 1 & -3\end{bmatrix}$ then $A$ is in the Echelon form

***Note :*** The rank of matrix in Echelon form is equal to the number of nonzero rows of the matrix.

**Ex :** $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is in echelon form number of non zero rows = 3

$\therefore$ The rank of A = 3

**Note :** The rank of $I_{nxn}$ is n

**4.** **Elementary transformations**

An elementary transformation is either of the following

**i)** Interchanging any two rows (or columns).

**ii)** Multiplication of elements of a row (or column) by a non zero number.

**iii)** Subtracting from (Adding to) the elements of one row, the corresponding elements of any other row multiplied by a non-zero number.

**5.** **EQUIVALENT MATRICES**

Two matrices $A$ and $B$ are called equaivalent if one can be obtained from the other by a finite number of elementary transformations. It is denoted by $A \sim B$.

**Note :** Elementary transformations donot change the rank of a matrix.

**i)** If '$r$' is rank of matrix of order $m \times n$ then $r \le \min(m, n)$.

**ii)** If A is a non-singular matrix of order n then $rank\left(A\right) = n$.

**iii)** If $A^T$ is a transpose of A, then $\text{rank}\left(A^T\right) = \text{rank}\left(A\right)$

**iv)** If $A^{\theta}$ is the transposed conjugate of A, then $rank\left(A^{\theta}\right) = rank\left(A\right)$.

**v)** $rank\left(A+B\right) \le rank\left(A\right) + rank\left(B\right)$.

**vi)** If A and B are two matrices such that the product AB is defined, then rank (AB) can not exceed the rank of the either matrix.

**6. Consistent and inconsistent systems**

The system of linear equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

**i)** If x = $\alpha$, y = $\beta$, z = $\gamma$ satisfying above system then x = $\alpha$, y = $\beta$, z = $\gamma$ is called solution of the system.

**ii)** The above system is said to be consistent if it possesses atleast one solution.

**iii)** The above system is said to be incosistent if it has no solution

**7. Augmented matrix**

Consider the equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$ in matrix form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ i.e. } AX = B$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is called coeffcient matrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ is called variable matrix}$$

$$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ is called constant matrix}$$

$K = [AB] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_1 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ is called augmented matrix

**8. Solution of system of non- homogenoeous linear equations in two unknowns :**

$a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$

Its matrix form is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\Rightarrow AX = B$

**i)** $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ system has unique solutions (consistent) $\Rightarrow X = A^{-1}B$

**ii)** $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ System has no solution (inconsistent)

**iii)** $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ System has infintely many solutions (consistent)

**9. Solution of system of non-homogeneous linear equations in three unknowns :**

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$ Its matrix form is

$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B$

**i)** $|A| \neq 0$ then the system has unique solution

**ii)** $|A| = 0$ then the system also has infinitely many solutions or no solution

**10. Solution of system of homogeneous linear equations in two unknowns :**

$a_1x + b_1y = 0$, $a_2x + b_2y = 0$. Its matrix form is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow AX = O$

**i)** $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ System has unique solution that is x = 0 = y (trivial solution)

**ii)** $\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow$ System has infinitely many solutions (non-trivial solutions)

**iii)** The above system of equations is always consistent.

**11. Solution of system of homogeneous linear equations in three unknowns :**

$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ Its matrix form is

$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow AX = O$

**i)** $|A| \neq 0$ then the system has unique solution i.e $x = y = z = 0$ (trivial solution)

**ii)** $|A| = 0$ then the system also has infinitely many solutions (non-trivial solutions)

**iii)** The above system of equations is always consistent.

**12. Solution of a system of Linear Equations by Matrix-Rank method :**

Let AX=B be a system of 'n' linear equations in 'n' variables.

1. Write the augmented matrix [A B]
2. Reduce the augmented matrix to echelon form using elementary row-operations
3. Determine the rank of the coefficient matrix A and augmented matrix [A B] by counting the number of non-zero rows in A and [A B].

**i)** If $rank(A) \neq rank(AB)$ ,then the system of equations is inconsistent.

**ii)** If $rank(A) = rank(AB)$ = the number of unknowns, then the system of equations is consistent and has a unique solution.

**iii)** If $rank(A) = rank(AB)$ < the number of un knowns, then the system of equations is consistent and has infinitely many solutions.

**13.** Let $AX = O$ be a homogeneous system of linear equations.

**a)** If $rank(A)$ = number of variables, then $AX = O$ have a trivial solution i.e. zero solution.

**b)** If $rank(A)$ < number of variables, then $AX = O$ have a non-trivial solution. It will have infinitely many solutions.

**14. Conditions for consistency of linear equations in three unknowns :**

The following cases may arise

**i)** If $\Delta \neq 0$ , or Rank (A) = Rank [AB] =3 then the system is consistent and has a unique solution, which is given by Cramer's rule:

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

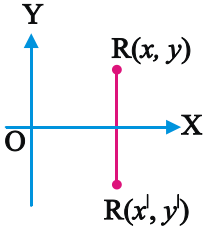
**ii)** If $\Delta = 0$ and at least one of the determinants $\Delta_1, \Delta_2, \Delta_3$ is non -zero,(or) $\Delta$ =0 and $(adjA)B \neq 0$ (or) rank (A) $\neq$ rank $[AB]$ the given system is inconsistent i.e., it has no solution.

**iii)** If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$ , (or) $\Delta = 0$ and $(adjA)B = 0$ (or) rank (A) = rank $[AB] < 3$ then the system is consistent and dependent, and has infinitely many solutions.

**15. Geometrical Transformations :**

**Reflection of a point $R(x, y)$ in x-axis :**

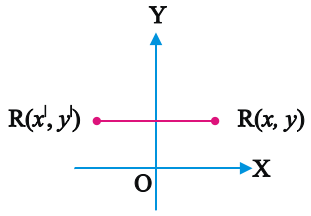
Let $R'(x', y')$ is the reflection of the point R $(x, y)$ on x-axis. The matrix of reflection is given by




$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

**Reflection of a point $R(x, y)$ in y-axis:**

$$R(x', y') = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



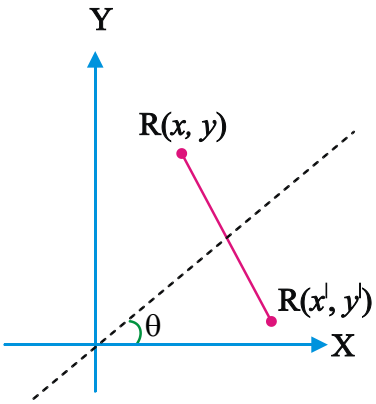

**Note:**

**i)** Reflection of a point $R(x, y)$ through origin is given by $\begin{bmatrix} x^1 \\ y^1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

**ii)** Reflection of point $R(x, y)$ in the line y = x is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**iii)** Reflection of point $R(x, y)$ in the line y = -x is given by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

**iv)** Reflection of point $R(x, y)$ in $y = (\tan\theta)x$ i.e $y = mx$ or $y = (\tan\theta)x$ *is*




given by $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} = \begin{pmatrix} \frac{1-\tan^2\theta}{1+\tan^2\theta} & \frac{2\tan\theta}{1+\tan^2\theta} \\ \frac{-2\tan\theta}{1+\tan^2\theta} & \frac{1-\tan^2\theta}{1+\tan^2\theta} \end{pmatrix} = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{-2m}{1+m^2} & \frac{1-m^2}{1+m^2} \end{pmatrix}$

**v)** Rotation of axes through an angle '$\theta$' is given by $R'(x', y') = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$