17 INVERSE TRIGONOMETRIC FUNCTIONS

SYNOPSIS

1. **Inverse of a function :** $f : A \to B$ is bijective $\Leftrightarrow f^{-1} : B \to A$ exists and it is also bijective. All trigonometric functions are not bijective functions. By restricting the domains of the functions,we make them bijective

i) The function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ defined by $f(x) = \sin x$ is a bijection.

Then $f^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is also a bijection. This function is called **inverse sine** function

and it is denoted by **Arc sine** x or $Sin^{-1} x$

ii) The function $f : [0,\pi] \to [-1,1]$ defined by $f(x) = \cos x$ is a bijection.

Then $f^{-1} : [-1,1] \to [0,\pi]$ is also a bijection. This function is called **inverse cosine** function and it is denoted by **Arc cos** x or $Cos^{-1} x$

iii) The function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ defined by $f(x) = \tan x$ is a bijection.

Then $f^{-1} : \mathrm{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is also a bijection. This function is called **inverse tangent** function

and it is denoted by **Arc tan** x or $Tan^{-1} x$

2. **Domains and Ranges of Inverse trigonometric functions :**

Function	Domain	Range
$Sin^{-1}x$	$[-1,1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$Cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$Tan^{-1}x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$Cot^{-1}x$	R	$(0,\pi)$
$Sec^{-1}x$	$(-\infty,-1] \cup [1,\infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$Cosec^{-1}x$	$(-\infty,-1] \cup [1,\infty)$	$\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

3. **i)** $\sin\theta = x \Leftrightarrow \sin^{-1} x = \theta,\ \forall x \in [-1,1], \theta = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

ii) $\cos\theta = x \Leftrightarrow \cos^{-1} x = \theta,\ \forall x \in [-1,1], \theta \in [0, \pi]$

iii) $\tan\theta = x \Leftrightarrow \tan^{-1} x = \theta,\ \forall x \in R, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

iv) $\cot\theta = x \Leftrightarrow \cot^{-1} x = \theta,\ \forall x \in R, \theta \in (0, \pi)$

v) $\sec\theta = x \Leftrightarrow \sec^{-1} x = \theta,\ \forall x \in (-\infty, -1] \cup [1, \infty), \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

vi) $\operatorname{cosec}\theta = x \Leftrightarrow \operatorname{cosec}^{-1} x = 0,\ \forall x \in (-\infty, -1] \cup [1, \infty), \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

4. **i)** $\sin(\sin^{-1}x) = x,\ \forall\ x \in [-1,1]$

ii) $\cos(\cos^{-1}x) = x,\ \forall\ x \in [-1,1]$

iii) $\tan(\tan^{-1}x) = x,\ \forall\ x \in (-\infty, \infty)$

iv) $\cot(\cot^{-1}x) = x,\ \forall\ x \in (-\infty, \infty)$

v) $\sec(\sec^{-1}x) = x,\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

vi) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x,\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

5. **i)** $\sin^{-1}(\sin\theta) = \theta,\ \forall\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

ii) $\cos^{-1}(\cos\theta) = \theta,\ \forall\ \theta \in [0, \pi]$

iii) $\tan^{-1}(\tan\theta) = \theta,\ \forall\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

iv) $\cot^{-1}(\cot\theta) = \theta,\ \forall\ \theta \in (0, \pi)$

v) $\sec^{-1}(\sec\theta) = \theta,\ \forall\ \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta,\ \forall\ \theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

6. **i)** $\sin^{-1}(-x) = -\sin^{-1}(x),\ \forall\ x \in [-1,1]$

ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x),\ \forall\ x \in [-1,1]$

iii) $\tan^{-1}(-x) = -\tan^{-1}(x),\ \forall\ x \in (-\infty, \infty)$

iv) $\cot^{-1}(-x) = \pi - \cot^{-1}(x),\ \forall\ x \in (-\infty, \infty)$

v) $\sec^{-1}(-x) = \pi - \sec^{-1}(x),\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x),\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

7. **i)** $\sin^{-1}(x) = \operatorname{cosec}^{-1}(1/x),\ \forall\ x \in [-1,0) \cup (0,1]$

ii) $\cos^{-1}(x) = \sec^{-1}(1/x),\ \forall\ x \in [-1,0) \cup (0,1]$

iii) $\tan^{-1}(x) = \cot^{-1}(1/x),\ \forall\ x \in (0, \infty)$ and $\tan^{-1}(x) = -\pi + \cot^{-1}(1/x),\ \forall\ x \in (-\infty, 0)$

iv) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right),\ \forall\ x \in (0, \infty) = \pi + \tan^{-1}\left(\frac{1}{x}\right),\ \forall\ x \in (-\infty, 0)$

v) $\sec^{-1} x = \cos ec^{-1}\left(\frac{1}{x}\right),\ \forall x \in (-\infty,-1]\cup[1,\infty)$

vi) $\cos ec^{-1}x = \sin^{-1}\left(\frac{1}{x}\right),\ \forall x \in (-\infty,-1]\cup[1,\infty)$

8. **i)** $\sin^{-1}x + \cos^{-1}(x) = \frac{\pi}{2},\ \forall\ x \in [-1,1]$

ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},\ \forall\ x \in (-\infty,\infty)$

iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2},\ \forall\ x \in (-\infty,-1]\cup[1,\infty)$

9. **i)**
$$Sin^{-1}x + Sin^{-1}y = \begin{cases} Sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} \\ if\ \ 0 \le x, y\ , x^2 + y^2 \le 1 \text{ or} \\ -1 \le x, y \le 1, xy < 0\ \ and\ x^2 + y^2 > 1 \\ \pi - Sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} \\ if\ 0 < x, y \le 1\ \ \text{and}\ \ x^2 + y^2 > 1 \\ -\pi - Sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} \\ \text{if } -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

ii)
$$Sin^{-1}x - Sin^{-1}y = \begin{cases} Sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\} \\ if\ \ 0 \le x, y < 1\ \text{and}\ x^2 + y^2 > 1 \text{ or} \\ -1 \le x, y \le 1, xy < 0 \text{ and } x^2 + y^2 \le 1 \\ \pi - Sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\} \\ if\ 0 < x \le 1, -1 \le y < 1 \text{and } x^2 + y^2 > 1 \\ -\pi - Sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\} \\ \text{if } 0 < y \le 1, -1 \le x < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

iii)
$$C\text{os}^{-1}x + C\text{os}^{-1}y = \begin{cases} C\text{os}^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1 \le x, y \le 1 \text{and } x + y \ge 0 \\ 2\pi - C\text{os}^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1 \le x, y \le 1 \text{and } x + y < 0 \end{cases}$$

iv)
$$C\text{os}^{-1}x - C\text{os}^{-1}y = \begin{cases} C\text{os}^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1 \le x, y \le 1, x \le y \\ -C\text{os}^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1 \le y \le 0, 0 < x \le 1 \text{and } x \ge y \end{cases}$$

10. **i)** $$Tan^{-1}x+Tan^{-1}y=\begin{cases}Tan^{-1}\left(\frac{x+y}{1-xy}\right) if\ x>0,y>0,xy<1\\ \pi+Tan^{-1}\left(\frac{x+y}{1-xy}\right) if\ x>0,y>0,xy>1\\ -\pi+Tan^{-1}\left(\frac{x+y}{1-xy}\right) if\ x<0,y<0,xy>1\\ \frac{\pi}{2}, if\ xy=1\end{cases}$$

ii) $$Tan^{-1}x-Tan^{-1}y=\begin{cases}Tan^{-1}\left(\frac{x-y}{1+xy}\right)\ if\ xy>-1\\ \pi+T\text{an}^{-1}\left(\frac{x-y}{1+xy}\right) if\ x>0,y<0 \text{ and } xy<-1\\ -\pi+T\text{an}^{-1}\left(\frac{x-y}{1+xy}\right) if\ x<0,y>0 \text{ and } xy<-1\end{cases}$$

11. **i)** If x, y, z have same sign and $xy+yz+zx<1$ then

$$Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=Tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

ii) $$\text{Tan}^{-1}\,x_1+\text{Tan}^{-1}\,x_2+.....\text{Tan}^{-1}\,x_n=\text{Tan}^{-1}\left(\frac{S_1-S_3+S_5-....}{1-S_2+S_4-S_6....}\right)$$

where $S_1=$ sum of values , $S_2=$ sum of product of taken two elements at a time and so on..,

$S_n=$ product of values.

12. **i)** $$Sin^{-1}x=\begin{cases}Cos^{-1}\sqrt{1-x^2} & if\ \ 0\le x\le 1\\ -Cos^{-1}\sqrt{1-x^2} & if\ \ -1\le x<0\\ Tan^{-1}\frac{x}{\sqrt{1-x^2}} & if\ \ x\in(-1,1)\end{cases}$$

ii) $$Cos^{-1}x=\begin{cases}Sin^{-1}\sqrt{1-x^2} & if\ \ x\in[0,1]\\ \pi-Sin^{-1}\sqrt{1-x^2} & if\ \ x\in[-1,0)\end{cases}$$

iii) $$Tan^{-1}x=\begin{cases}Sin^{-1}\frac{x}{\sqrt{1+x^2}} & for\ \ x>0\\ Cos^{-1}\frac{1}{\sqrt{1+x^2}} & for\ \ x>0\end{cases}$$

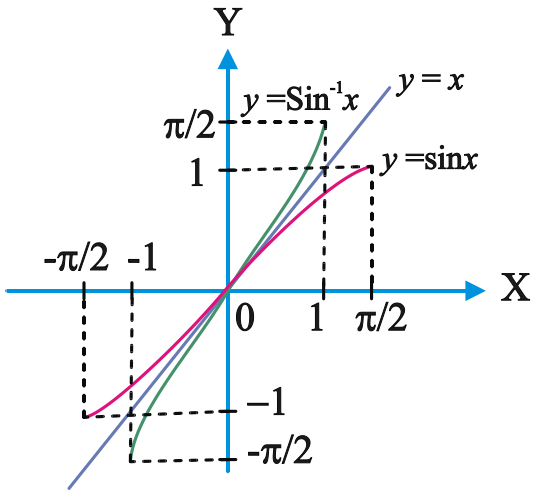
13. **i)** $$2Tan^{-1}x=\begin{cases} Tan^{-1}\left(\frac{2x}{1-x^2}\right) & if\ -1<x<1 \\ \pi+Tan^{-1}\left(\frac{2x}{1-x^2}\right) & if\ x>1 \\ -\pi+Tan^{-1}\left(\frac{2x}{1-x^2}\right) & if\ x<-1 \\ \frac{\pi}{2} & if\quad x=1 \\ -\frac{\pi}{2} & if\quad x=-1 \end{cases}$$

ii) $$2Tan^{-1}x=\begin{cases} -\pi-Sin^{-1}\left(\frac{2x}{1+x^2}\right) & if\ x<-1 \\ Sin^{-1}\left(\frac{2x}{1+x^2}\right) & if\ -1\le x\le 1 \\ \pi-Sin^{-1}\left(\frac{2x}{1+x^2}\right) & if\ x>1 \end{cases}$$

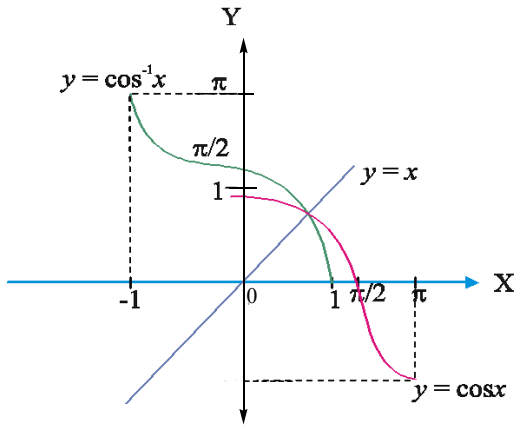
iii) $$2Tan^{-1}x=\begin{cases} -Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & if\ -\infty<x\le 0 \\ Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & if\ 0\le x<\infty \end{cases}$$

14. **Graphs of inverse circular functions:**

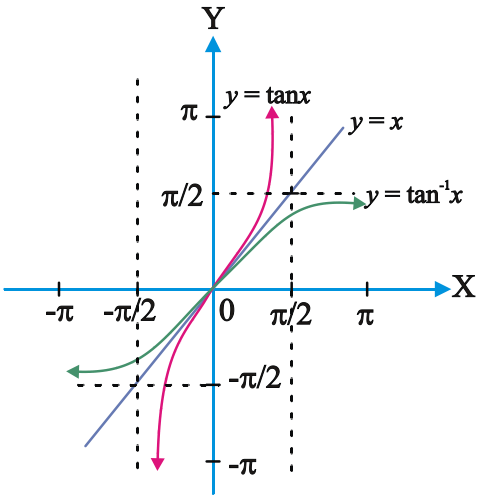
i) $y = \text{Sin}^{-1} x, |x| \le 1, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



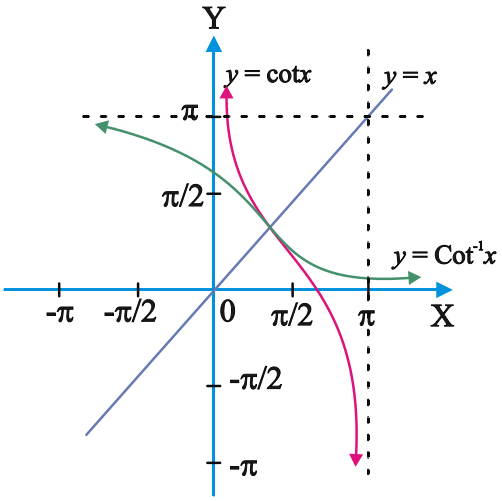
ii) $y = \text{Cos}^{-1} x, |x| \le 1, y \in [0, \pi]$



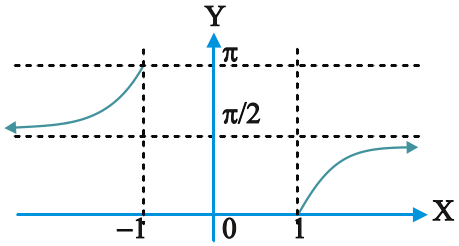
iii) $y = \text{Tan}^{-1}x\ ,\ x \in R, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



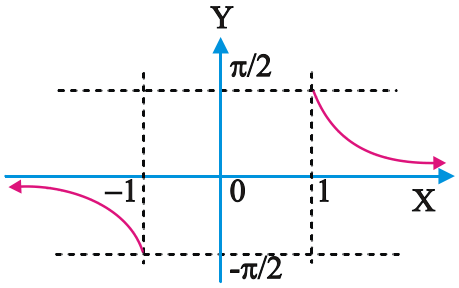
iv) $y = \text{Cot}^{-1}x\ ,\ x \in R, y \in (0\ ,\ \pi)$



v) $y = \text{Sec}^{-1} x, |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

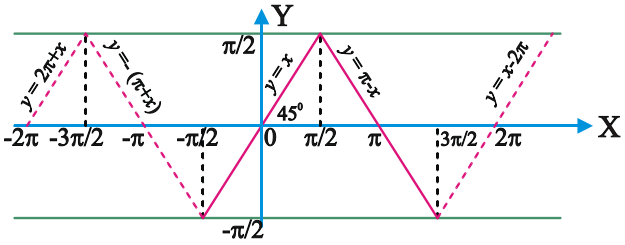


vi) $y=\text{Cosec}^{-1}x, \ |x| \geq 1, \ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

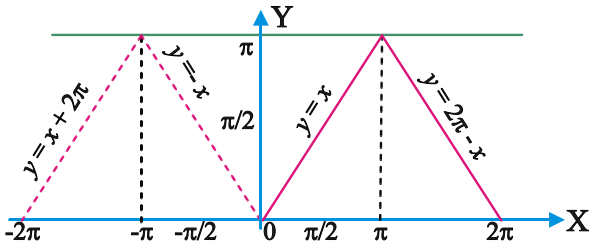


15. Some useful periodic graphs:

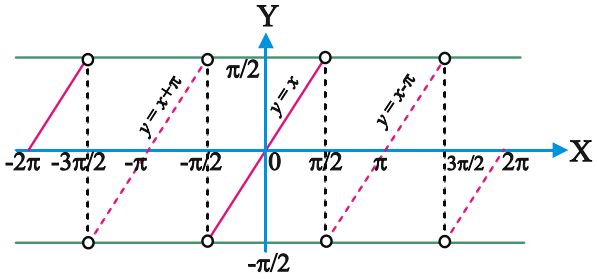
i) $y = Sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\frac{3\pi}{2} \leq x \leq -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ -2\pi + x, & \frac{3\pi}{2} \leq x \leq \frac{5\pi}{2} \end{cases}$ y is Periodic with period 2π and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



ii) $y = Cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi \leq x \leq 2\pi \\ -2\pi + x, & 2\pi \leq x \leq 3\pi \end{cases}$ y is periodic with period 2π and $y \in [0, \pi]$

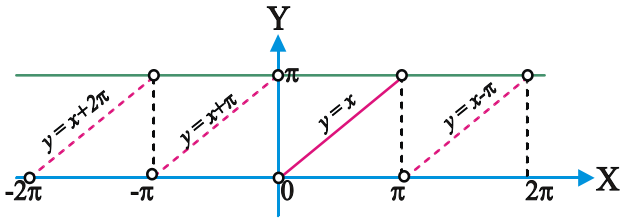


iii) $y = Tan^{-1}(\tan x) = \begin{cases} x+\pi, & -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -\pi + x, & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} < x < \frac{5\pi}{2} \end{cases}$ y is periodic with period π and $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

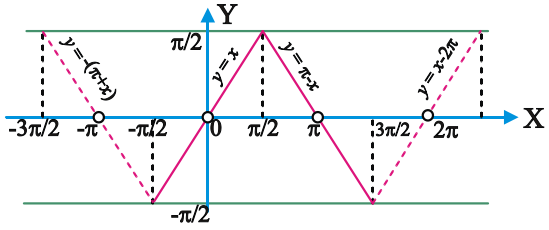


iv) $y = \text{Cot}^{-1}(\text{Cot}\,x) = x, \quad x \in (0, \pi)$ and so on.

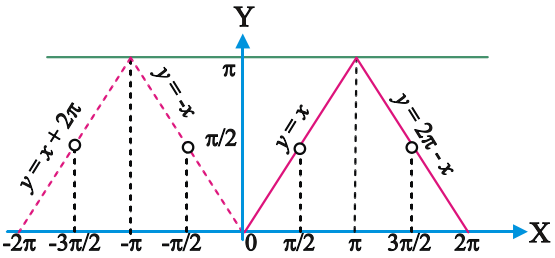
y is periodic with period π and $y \in (0, \pi)$



v) $y = Cosec^{-1}(\cos ecx) = \begin{cases} x, & x \in \left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\ \pi - x, & x \in \left[\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right] \\ \textit{and so on} \end{cases}$ y is periodic with period 2π *and* $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

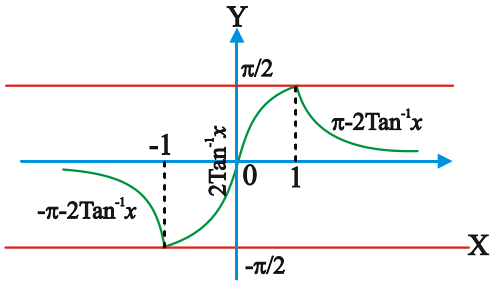


vi) $y = Sec^{-1}(\sec x) = \begin{cases} x, & x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ 2\pi - x, & x \in \left[\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] \\ \textit{and so on} \end{cases}$ y is periodic with period 2π and $y \in [0, \pi]$

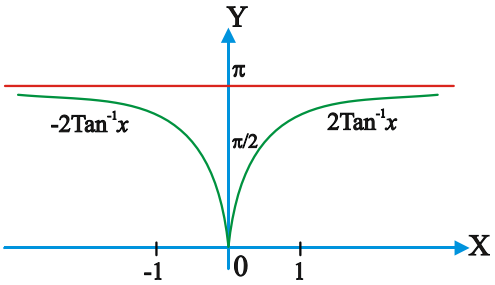


16. Transformation of Inverse functions by elementry substitution and their graphs:

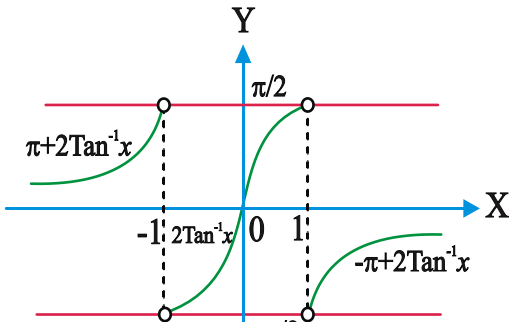
i) $\text{Sin}^{-1}\frac{2x}{1+x^2} = \begin{cases} -\pi - 2\,\text{Tan}^{-1}x & x \le -1 \\ 2\,\text{Tan}^{-1}x & -1 \le x \le 1 \\ \pi - 2\,\text{Tan}^{-1}x & x \ge 1 \end{cases}$



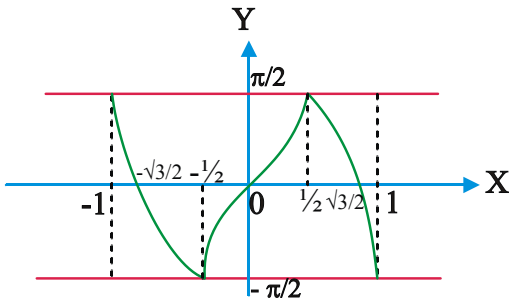
ii) $\text{Cos}^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\,\text{Tan}^{-1}x & x \ge 0 \\ -2\,\text{Tan}^{-1}x & x < 0 \end{cases}$



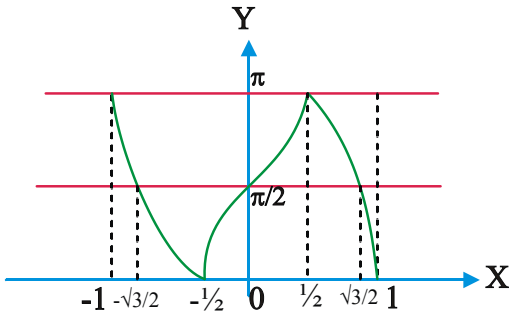
iii) $\text{Tan}^{-1}\frac{2x}{1-x^2} = \begin{cases} \pi + 2\,\text{Tan}^{-1}x & x < -1 \\ 2\,\text{Tan}^{-1}x & -1 < x < 1 \\ -\pi + 2\,\text{Tan}^{-1}x & x > 1 \end{cases}$



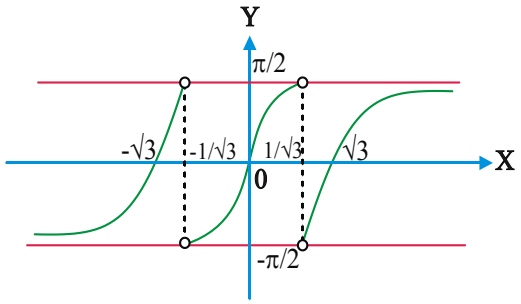
iv) $Sin^{-1}\left(3x-4x^3\right)=\begin{cases}-\left(\pi+3Sin^{-1}x\right) & if\ -1\le x\le -1/2\\ 3Sin^{-1}x & if\ -1/2\le x\le 1/2\\ \pi-3Sin^{-1}x & if\ \ 1/2\le x\le 1\end{cases}$



v) $\text{Cos}^{-1}\left(4x^3-3x\right)=\begin{cases}3\cos^{-1}x-2\pi & if\ -1\le x\le -1/2\\ 2\pi-3\text{Cos}^{-1}x & if\ -1/2\le x\le 1/2\\ 3C\text{os}^{-1}x & if\ \ 1/2\le x\le 1\end{cases};$



vi) $\text{Tan}^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)=\begin{cases}3\,\text{Tan}^{-1}x & if\ -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\\ \pi+3\,\text{Tan}^{-1}x & if\ x>\dfrac{1}{\sqrt{3}}\\ -\pi+3\,\text{Tan}^{-1}x & if\ x<-\dfrac{1}{\sqrt{3}}\end{cases}$



17. Some important facts:

i) $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\frac{\pi}{2},\ if\ \ xy+yz+zx=1$

ii) $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\pi,\ \ if\ \ x+y+z=xyz$

iii) $Tan^{-1}\frac{a}{x}+Tan^{-1}\frac{b}{x}=\frac{\pi}{2}$,then $x=\sqrt{ab}$

iv) $Sin^{-1}\frac{a}{x}+Sin^{-1}\frac{b}{x}=\frac{\pi}{2}$, then $x=\sqrt{a^2+b^2}$

v) $Tan^{-1}\left(\frac{p}{q}\right)+Tan^{-1}\left(\frac{q-p}{q+p}\right)=\frac{\pi}{4}$

vi) $\mathrm{Tan}^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x\ \ \ if\ \ x<1$

vii) $\mathrm{Tan}^{-1}\left(\frac{1-x}{1+x}\right)=\frac{\pi}{4}-\tan^{-1}x\ \ \ if\ \ x>-1$

viii) If $Tan^{-1}x+Tan^{-1}y=\frac{\pi}{2}$ then xy=1.

ix) $Cot^{-1}x+Cot^{-1}y=\frac{\pi}{2}$ then xy=1

x) If $\mathrm{Cos}^{-1}\mathrm{x}+\mathrm{Cos}^{-1}\mathrm{y}+\mathrm{Cos}^{-1}\mathrm{z}=3\pi$ then $xy+yx+zx=3$

xi) If $\mathrm{Sin}^{-1}\mathrm{x}+\mathrm{Sin}^{-1}\mathrm{y}+\mathrm{Sin}^{-1}\mathrm{z}=\frac{3\pi}{2}$ then $xy+yx+zx=3$

xii) If $\mathrm{Sin}^{-1}\mathrm{x}+\mathrm{Sin}^{-1}\mathrm{y}=\theta$ then $\mathrm{Cos}^{-1}\mathrm{x}+\mathrm{Cos}^{-1}\mathrm{y}=\pi-\theta$

xiii) If $\mathrm{Cos}^{-1}\mathrm{x}+\mathrm{Cos}^{-1}\mathrm{y}=\theta$ then $\mathrm{Sin}^{-1}\mathrm{x}+\mathrm{Sin}^{-1}\mathrm{y}=\pi-\theta$

xiv) If $\mathrm{aSin}^{-1}\mathrm{x}-\mathrm{bCos}^{-1}\mathrm{x}=\mathrm{c}$ then $a\mathrm{Sin}^{-1}x+b\mathrm{Cos}^{-1}x=\frac{\pi ab+c(a-b)}{a+b}$

xv) If $Cos^{-1}\frac{x}{a}+Cos^{-1}\frac{y}{b}=\theta$ then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=Sin^2\theta$

xvi) If $Sin^{-1}\frac{x}{a}+Sin^{-1}\frac{y}{b}=\theta \quad then \ \frac{x^2}{a^2}+\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=Sin^2\theta$

xvii) $Tan^{-1}\left(\frac{1}{1+x(x+1)}\right)+Tan^{-1}\left(\frac{1}{1+(x+1)(x+2)}\right)+.....+Tan^{-1}\left(\frac{1}{1+(x+n-1)(x+n)}\right)$

$=Tan^{-1}(x+n)-Tan^{-1}x, n\in N$

xviii) $Tan^{-1}\frac{1}{2}+Tan^{-1}\frac{1}{3}=\frac{\pi}{4}$

xix) $Tan^{-1}2+Tan^{-1}3=\frac{3\pi}{4}$

18. If an expression contains

i) $\sqrt{a^2-x^2}, put\ x=a\sin\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

ii) $\sqrt{a^2+x^2}, put\ x=a\tan\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

iii) $\sqrt{x^2-a^2}, put\ x=a\sec\theta, \theta\in(0,\pi)$

19. Range of some special inverse Trigonometric Functions :

i) $\frac{\pi^3}{32}\le\left(\text{Sin}^{-1}x\right)^3+\left(\text{Cos}^{-1}x\right)^3\le\frac{7\pi^3}{8}$

ii) $\frac{\pi^2}{8}\le\left(\text{Sin}^{-1}x\right)^2+\left(\text{Cos}^{-1}x\right)^2\le\frac{5\pi^2}{4}$

iii) $-\frac{\pi^2}{4}\le\left(\text{Cos}^{-1}x\right)^2-\left(\text{Sin}^{-1}x\right)^2\le\frac{3\pi^2}{4}$