18 HYPERBOLIC FUNCTIONS

SYNOPSIS

1. **i)** The exponential function, e^x for $x \in R$ is defined as $e^x = 1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} + ... + \frac{x^n}{\angle n} + ...\infty$

$$= \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \right)$$

ii) For $x \in R$, $e^{-x} = 1 - \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} + ... + (-1)^n \frac{x^n}{\angle n} + ...\infty = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{\angle n}$

2.

Hyperbolic Function	Domain	Range
$\sinh(x) = \frac{e^x - e^{-x}}{2}$	R	R
$\cosh(x) = \frac{e^x + e^{-x}}{2}$	R	$[1, \infty)$
$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	R	$(-1, 1)$
$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$	$R - \{0\}$	$R - [-1, 1]$ (or) $(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$	R	$(0, 1]$
$\operatorname{cosec h}(x) = \frac{2}{e^x - e^{-x}}$	$R - \{0\}$	$R - \{0\}$

3. **i)** $\sinh(-x) = -\sinh(x)$

ii) $\cosh(-x) = \cosh(x)$

iii) $\tanh(-x) = -\tanh(x)$

iv) $\coth(-x) = -\coth(x)$

v) $\operatorname{sech}(-x) = \operatorname{sech}(x)$

vi) $\operatorname{cosech}(-x) = -\operatorname{cosech}(x)$

4. **i)** $\cosh^2(x) - \sinh^2(x) = 1$

ii) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

iii) $\coth^2 x - 1 = \operatorname{cosec}^2 x$

5. **i)** $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

ii) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

iii) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

iv) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

v) $\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$

vi) $\tanh(x-y) = \frac{\tanh(x) - \tanh(y)}{1 - \tanh(x)\tanh(y)}$

vii) $\text{Coth}(x+y) = \dfrac{\text{Cot}h(x).\text{Cot}h(y)+1}{\text{Cot}hy+\text{Cot}hx}$

viii) $\text{Coth(x-y)} = \dfrac{\text{Cot}hx.\text{Cot}hy-1}{\text{Cot}hy-\text{Cot}hx}$

6. **i)** $\text{Sinh}(2x)=2\sinh(x)\cosh(x) = \dfrac{2\tanh x}{1-\tanh^2 x}$

ii) $\text{Cosh}(2x) = \text{Cosh}^2(x)+\text{Sinh}^2(x) = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

iii) $\text{Tanh}(2x) = \dfrac{2\text{Tan}h(x)}{1+\text{Tan}h^2(x)}$

iv) $\text{Sinh}(3x) = 3\sinh(x) + 4\sinh^3(x)$

v) $\text{Cosh}(3x) = 4\cosh^3(x) - 3\text{Cosh}(x)$

vi) $\text{Tanh}(3x) = \dfrac{3\tanh(x)+\tanh^3(x)}{1+3\tanh^2(x)}$

7. **i)** $(\cosh x+\sinh x)^n = \cosh nx+\sinh nx = e^{nx}$

ii) $(\cosh x-\sinh x)^n = \cosh nx-\sinh nx = e^{-nx}$

iii) $\text{Cosh } 2nx + \text{Sinh } 2nx = \left(\dfrac{1+\text{Tan}hx}{1-\text{Tan}hx}\right)^n$

8. $\text{Sinh}x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\ldots\ldots\ldots\ldots \infty$

$\text{cosh}x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\ldots\ldots\ldots\ldots \infty$

9. $\sinh(x+y) + \sinh(x-y) = 2\sinh x\cosh y$

$\sinh(x+y) - \sinh(x-y) = 2\cosh x\sinh y$

$\cosh(x+y) + \cosh(x-y) = 2\cosh x\cosh y$

$\cosh(x+y) - \cosh(x-y) = 2\sinh x\sinh y$

$\sinh(x+y)\sinh(x-y) = \sinh^2 x-\sinh^2 y = \cosh^2 x-\cosh^2 y$

$\cosh(x+y)\cosh(x-y) = \cosh^2 x+\sinh^2 y = \sinh^2 x+\cosh^2 y$

10. $\sinh x+\sinh y = 2\sinh\left(\dfrac{x+y}{2}\right)\cosh\left(\dfrac{x-y}{2}\right)$

$\sinh x-\sinh y = 2\cosh\left(\dfrac{x+y}{2}\right)\sinh\left(\dfrac{x-y}{2}\right)$

$\cosh x+\cosh y = 2\cosh\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

$\cosh x-\cosh y = 2\sinh\left(\dfrac{x+y}{2}\right)\sinh\left(\dfrac{x-y}{2}\right)$

11. Logarithmic form of inverse hyperbolic functions

i) $\sinh^{-1} x = \log_e\left(x+\sqrt{x^2+1}\right), \forall x \in R$

ii) $\cosh^{-1} x = \log_e\left(x+\sqrt{x^2-1}\right), \forall x \in [1,\infty)$

iii) $\tan h^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \forall x \in (-1,1)$

iv) $\coth^{-1} x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right), \forall x \in (-\infty,-1)\cup(1,\infty)$

v) $\sec h^{-1} x = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right), \forall x \in (0,1]$

vi) $\cos ec\, h^{-1} x = \log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right), \forall x > 0 \;= \log\left(\frac{1-\sqrt{1+x^2}}{x}\right), \forall x < 0$

12.

Function	Domain	Range
$\sinh^{-1}(x)$	R	R
$\cosh^{-1}(x)$	$[1,\infty)$	$[1,\infty)$
$\tanh^{-1}(x)$	$(-1,1)$	R
$\coth^{-1}(x)$	$R-[-1,1]$	$R-\{0\}$
$\text{sech}^{-1}(x)$	$(0,1]$	$[1,\infty)$
$\cos ec\, h^{-1}(x)$	$R-\{0\}$	$R-\{0\}$

13. **i)** $\text{Sinh}^{-1}(x) = \cosh^{-1}(\sqrt{x^2+1}) = \text{cosech}^{-1}\left(\frac{1}{x}\right) = \text{Tanh}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

ii) $\text{Cosh}^{-1}(x) = \sinh^{-1}(\sqrt{x^2-1}) = \text{sech}^{-1}\left(\frac{1}{x}\right) = \text{Tanh}^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

14. $\sinh x$, $\tanh x$ are an odd functions and $\sinh^{-1}x$, $\text{Tanh}^{-1}x$ are also odd functions. $\cosh x$ is an even function but $\text{Cosh}^{-1}x$ is neither even nor odd function.