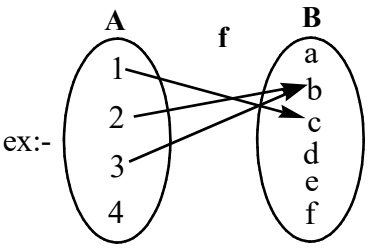
FUNCTIONS

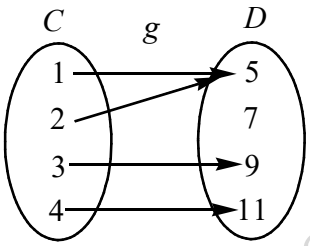
SYNOPSIS

1. **Cartesian Product of two sets:** Let A, B are two sets then cartesian product of A, B is denoted by A x B (Read it as "A cross B") and is defined as A x B = $\{(a,b)/a \in A, b \in B\}$
2. **Relation :-** Let A, B are two sets then every subset of A x B is a relation from A to B
   ex:- A = {1, 2, 3}, B = {a, b}
   A x B = {(1, a), (1, b),(2, a),(2, b),(3, a), (3, b)}
   R = {(1, a),(1, b),(2, a)} is a relation from A to B
3. **Function:-** Each and every element in A is associated to a unique element in B then the Relation f from A to B is said to be a function from A to B. It is denoted by $f: A \to B$ (Read it as f maps from A to B)

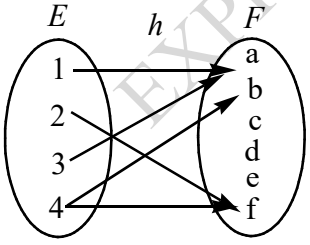


ex:-

$f: A \to B$ is not a function



$g: C \to D$ is a function



$h: E \to F$ is not a function

Note: The number of functions from A to B is $\{n(B)\}^{n(A)}$

4. **Image and Pre-Image:-** If $f: A \to B$ is a function and $(a, b) \in f$ where $a \in A$, $b \in B$ then 'b' is called image of a and a is called pre - image of b.
5. **Domain, Co-Domain, Range:-** Let $f: A \to B$ is a function then A is called domain of 'f' and B is called Co-Domain of 'f' and set of all f images of the elements of A is called range of f, it is denoted by f(A).

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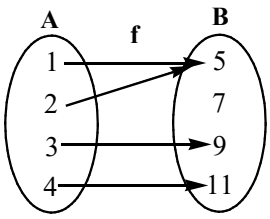
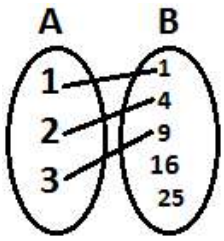
Ex :-



f : A → B is a function
Domain = A = {1, 2, 3,4},
Co - Domain = B = {5, 7, 9, 11},
Range = f(A) = {5, 9, 11}
Ex : Let A = {1, 2, 3}, B = {1, 4, 9, 16, 25} and relation f from A to B is defined as
f(x) = x^2 is function (or) not $f(x) = x^2$



$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$f : A \to B$ is a function, Domain = {1,2,3}
Co-Domain = {1, 4,9,16,25}, Range {1,4,9}

Ex:- A Relation from R → R is defined as $f(x) = x^2$ is a function

Ex:- A Relation from R → R is defined as $f(x) = \sqrt{x}$ is not a function

$\because -1 \in R$ (Domain), $f(-1) = \sqrt{-1} \notin R$(Codomain)
-1 has no image in R, $\therefore$ f is not a function

6. Polynomial function: If f: $f : R \to R$ is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + .. + a_{n-1}x + a_n$,

$a_0 \neq 0, a_1, a_2$ ..... $a_n \in R$, n is a non-negative integer is a polynomial function of degree 'n' in x

E.g: i) $x^4 - x + 2$ is a polynomial function.

ii) $x^4 - \sqrt{x} + 2$ is not a polynomial function

Note : If the domain of a function is not given then its domain is taken as the set of all real values of x for which f(x) is define.

Ex : $f(x) = x^2 + 3x + 5$, domain = R

7. Constant function: A function $f : A \to B$ is a constant function if the range of f contains only one element.

Note : i) The number of constant functions from A to B is n(B).

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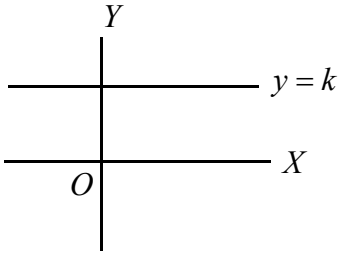
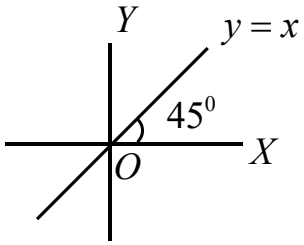
ii) The graph of constant function is a line parallel to x-axis.



iii) Range of any constant function is a singleton set

8. **Identity function:** Let A be a non - empty set then $f : A \to A$ defined by $f(x) = x, \forall\ x \in A$ is called the identity function on A and it is denoted by I_A.

Note : The graph of identity function is a straight line passing through origin and inclined at an angle of 45^0 with x-axis.



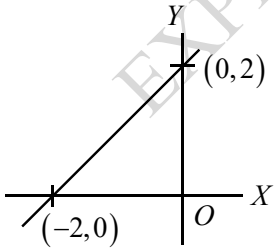
9. **Linear Function :** $y = f(x) = ax + b,\ a \neq 0$ is a linear function.It represents a straight line.

Ex : y = x + 2

y = 0, x = -2 $\to$ x - intercept

x = 0, y = 2 $\to$ y - intercept

given line passing through (-2, 0), (0, 2)



10. **Quadratic function :**

Quadration equation :- An equation in the form $ax^2 + bx + c = 0$ is called 'a'quadratic equation in x Where, $a \neq 0, b, c$ are complex numbers and a,b,c are called coefficients of the quadratic equation

In fact $y = ax^2 + bx + c, a \neq 0$ is called a **quadratic function**

Nature of the roots :- The roots of $ax^2 + bx + c = 0$ $are \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Let $\Delta = b^2 - 4ac$ and if a, b, c are real coefficients then

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i) $\Delta > 0 \Rightarrow$ roots are real and distinct

ii) $\Delta = 0 \Rightarrow$ roots are real and equal

iii) $\Delta < 0 \Rightarrow$ roots are complex (Imaginary)

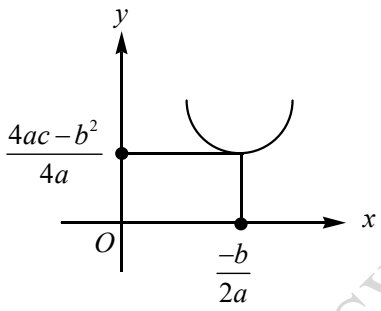
Graph of quadratic expression

i) If $\boxed{a > 0}$ then

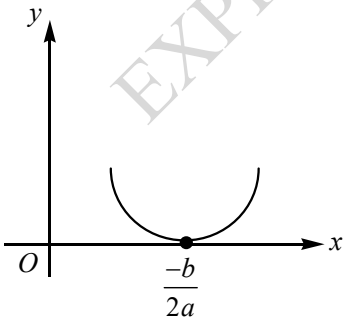
$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ represents an upward parabola. It attains minimum value at $x = \frac{-b}{2a}$ and that minimum value $= \frac{4ac - b^2}{4a}$.

Vertex $= \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

$\forall x \in R$, Range $= \left[\frac{4ac - b2}{4a}, \infty\right)$



$\boxed{\Delta < 0}$



$\boxed{\Delta = 0}$

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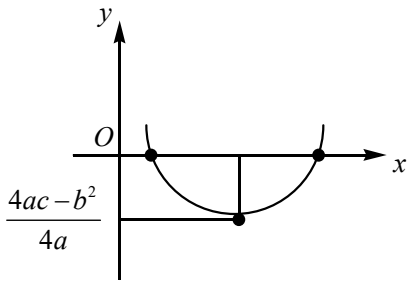


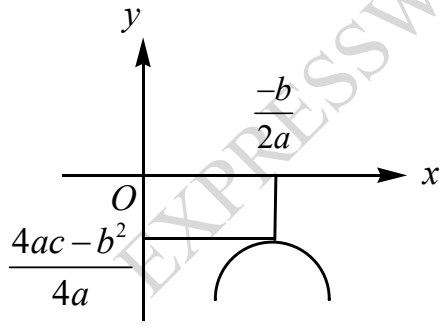
$\boxed{\Delta > 0}$

ii) If $\boxed{a < 0}$ then

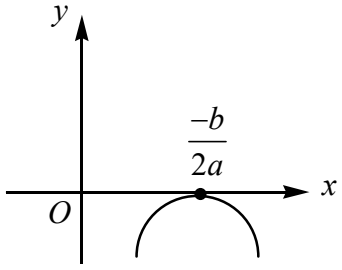
$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ represents a downward parabola. It attains maximum value at $x = \frac{-b}{2a}$ and that maximum value $= \frac{4ac - b^2}{4a}$

vertex $= \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

$\forall x \in R$, Range $= \left(-\infty, \frac{4ac - b^2}{4a}\right]$



$\boxed{\Delta < 0}$



$\boxed{\Delta = 0}$

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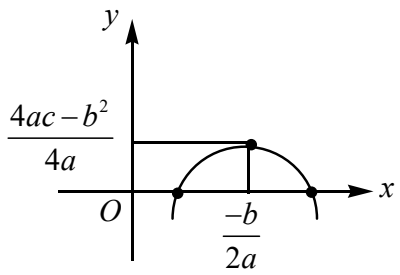
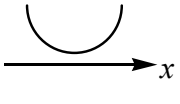
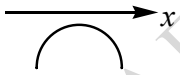
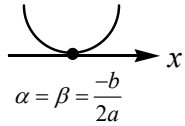
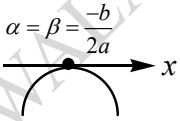
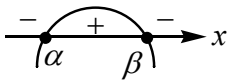


$\boxed{\Delta > 0}$

Solving Quadratic inequation

let $y = f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$

i)

	$a > 0$	$a < 0$	Conclusion on $f(x)$
$\Delta < 0$			The roots of f(x) = 0 are imaginary then $\forall x \in R$, f(x) and 'a' will have the same sign
$\Delta = 0$			The roots of f(x) = 0 are real and equal to $\alpha = \frac{-b}{2a}$ then f(x) is a perfect square i.e. $f(x) = a\left(x + \frac{b}{2a}\right)^2$ and $\forall x \in R - \left\{\frac{-b}{2a}\right\}, f(x)$ and a will have the same sign
$\Delta > 0$			The roots α, β of f(x) = 0 are real and distanct (i) $\alpha < x < \beta$ then f(x) and 'a' will have opposite in signs (ii) $x < \alpha$ (or) $x > \beta$ then $f(x)$ and 'a' will have the same sign

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ii)

	$\Delta<0$		$\Delta=0$		$\Delta>0$	
	$(a>0)$	$(a<0)$	$(a>0)$	$(a<0)$	$(a>0)$	$(a<0)$
$f(x)=0$	ϕ	ϕ	$\left\{\frac{-b}{2a}\right\}$	$\left\{\frac{-b}{2a}\right\}$	$\{\alpha,\beta\}$	$\{\alpha,\beta\}$
$f(x)>0$	R	ϕ	$R-\left\{\frac{-b}{2a}\right\}$	ϕ	$(-\infty,\alpha)\cup(\beta,\alpha)$	(α,β)
$f(x)\geq 0$	R	ϕ	R	$\left\{\frac{-b}{2a}\right\}$	$(-\infty,\alpha]\cup[\beta,\alpha)$	$[\alpha,\beta]$
$f(x)<0$	ϕ	R	ϕ	$R-\left\{\frac{-b}{2a}\right\}$	(α,β)	$(-\infty,\alpha)\cup(\beta,\infty)$
$f(x)\leq 0$	ϕ	R	$\left\{\frac{-b}{2a}\right\}$	R	$[\alpha,\beta]$	$(-\infty,\alpha]\cup[\beta,\infty)$

Note : i) : (x - a)(x - b)< 0 Þ x Î (a , b) $-\infty$ + α − β + ∞

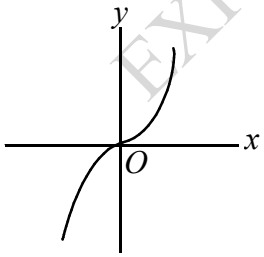
ii) (x - a)(x - b)£ 0 Þ x Î [a , b]

iii) (x - a)(x - b)> 0 Þ x Î (- ¥ ,a)È (b,¥)

iv) (x - a)(x - b)³ 0 Þ x Î (- ¥ ,a)È (b,¥), Where $\alpha<\beta$

11. **Cubic function :** An equation of the form $f(x)=ax^3+bx^2+cx+d\ (a\neq 0)$ is called a cubic function

Ex : $f(x)=x^3$



Domain = R, Range = R

12. **Rational Function:** A function of the form $\frac{f(x)}{g(x)}$ where f(x) and g(x) are polynomial functions and $g(x)\neq 0$ is called a rational function.

Ex 1 : $f(\text{x})=\frac{x-3}{x-6}$, Domain = R - {6},

Ex 2 : $f(\text{x})=\frac{1}{x}$

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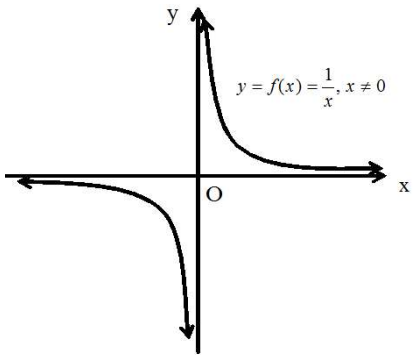
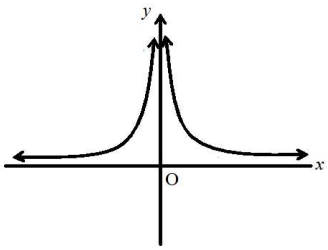


Domain = R – {0}, Range = R – {0}

Ex 3 : $f(x) = \frac{1}{x^2}$



Domain = R – {0}, Range = $(0, \infty)$

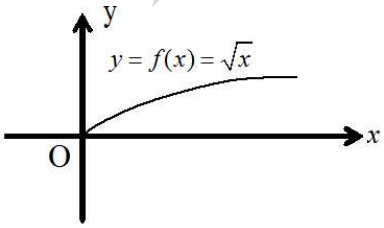
13. Irrational functions :

Functions involving radical sign (or) fractional power of independent variable are called irrational functions

Ex : $\sqrt{x}, \frac{2x^2 + \sqrt{x}}{\sqrt{1+5x^2}}$

Graph of $\sqrt{x}$

Domain = $[0, \infty)$, Range = $[0, \infty)$



14. Absolute Value Function (Modulus Function) : The function $f: R \to R$ defined by $f(x) = |x|$ is called the mod function or modulus function or absolute value function. Dom $f = R$, Range $f = [0, \infty)$ The piece - wise definition of the mod function is

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

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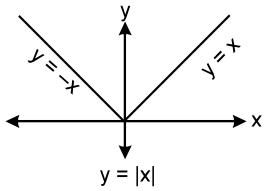
The graph of $y = f(x) = |x|$ is



Properties of modulus :-

i) $|x| \geq 0 \ \forall x \in R$

ii) $|x| = 0 \Rightarrow x = 0$

iii) $|x| > x, x < 0$

iv) $|x| < x$, No real x

v) $|-x| = |x|$

vi) $|x^n| = |x|^n$

vii) $|xy| = |x||y|$

viii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

ix) $|x+y| \leq |x| + |y|$

x) $|x+y| = |x| + |y|$, for $xy \geq 0$

xi) $|x+y| < |x| + |y|$, for $xy < 0$

xii) $||x| - |y|| \leq |x+y| \leq |x| + |y|$

$||x| - |y|| \leq |x-y| \leq |x| + |y|$

xiii) $|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$

$|x| \leq a \Rightarrow -a \leq x \leq a \Rightarrow x \in [-a, a]$

$|x| > a \Rightarrow x < -a \text{ (or) } x > a \Rightarrow x \in (-\infty, -a) \cup (a, \infty)$

$|x| \geq a \Rightarrow x \leq -a \text{ (or) } x \geq a \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

15. Greatest Integer function (or) step function (or) floor function :-

If $x \in R$ and $n \leq x < n+1, n \in Z$ then

f(x) = n is called step function. It is denoted by [x] and read it as "step x".

Ex:- [1.5] = 1, [-1.5] = -2, [1.2] = 1, [-1.2] = -2, [1] = 1, [-1] = -1

Note:-

$\forall x \in [0,1) \Rightarrow [x] = 0$

$\forall x \in [-1, 0) \Rightarrow [x] = -1$

$\forall x \in [1, 2) \Rightarrow [x] = 1$

$\forall x \in [2, 3) \Rightarrow [x] = 2$

$\forall x \in [-2, -1) \Rightarrow [x] = -2$

Note:- f(x) = [x] is defined $\forall x \in R$

Domain = R, Range = Z,

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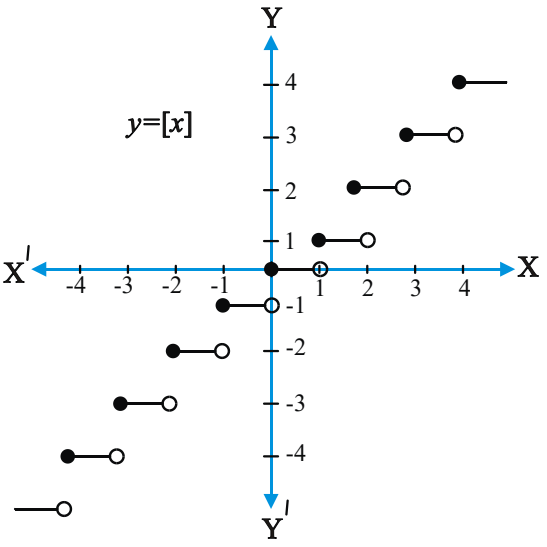
Graph of [x]:-



Properties

i) $[x] \le x, \forall x \in R$

ii) $[x] = x,\ x \in Z$

iii) $[x] < x,\ x \in R - Z$

iv) $[x] > x$, No real x

v) $x \in R,\ K \in Z$ then $[x + k] = [x] + k$

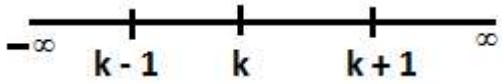
vi) $[-x] = -[x],\ \forall x \in Z$

vii) $[-x] = -[x]-1,\ \forall x \in R - Z$

viii) $[x] + [-x] = 0,\ \forall x \in Z$

$[x] + [-x] = -1,\ \forall x \in R - Z$

ix) $k \in Z,\ [x] \ge k \Rightarrow x \in [k, \infty)$



x) $K \in z, [x] > k \Rightarrow x \in [k+1, \infty)$

xi) $K \in z, [x] \le k \Rightarrow x \in (-\infty, k+1)$

xii) $K \in z, [x] < k \Rightarrow x \in (-\infty, k)$

xiii) $[[x]] = [x], \forall\ x \in R$

xiv) $x - 1 < [x] \le x, \forall\ x \in R$

xv) $[x + y] = [x] + [y + x - [x]], \forall\ x, y \in R$

xvi) $[x + y] \ge [x] + [y]$ where $x, y \in R$

xvii) $[x + y] = [x] + [y]$ for $x, y \in R$, $\{x\} + \{y\} < 1$ where $\{x\}$ is denotes fractional parts of x, $\{y\}$ is denotes fractional part of y

xviii) $\left[\frac{x}{n}\right] + \left[\frac{x+1}{n}\right] + \left[\frac{x+2}{n}\right] + ... + \left[\frac{x+n-1}{n}\right] = [x], n \in N$

xix) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + ... + \left[x + \frac{n-1}{n}\right] = [nx]$

16. Least Integer Function:

If $x \in R$ then the least integer greater than or equal to x is denoted by (x) is called least integer function or ceiling function.

Eg : i) (3.2) = 4,

ii) (-2.3) = -2,

iii) (5) = 5

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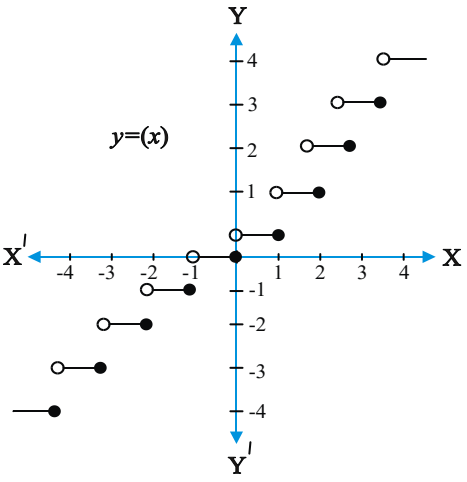

Note :

i) $(x)=\begin{cases}-1, & -2<x\le -1\\ 0, & -1<x\le 0\\ 1, & 0<x\le 1\\ 2, & 1<x\le 2\end{cases}$

ii) Domain of (x) is R and Range is Z.

Properties of L.I.F

i) $((x))=x$

ii) $(x)=\begin{cases}[x]=x;\ x\in I\\ [x]+1;\ x\notin I\end{cases}$

iii) $(x)+(-x)=\begin{cases}0;\ x\in I\\ 1;\ x\notin I\end{cases}$

iv) $(x)-(-x)=\begin{cases}2x;\ x\in I\\ 2(x)-1;\ x\notin I\end{cases}$

v) $(x)=\left(\frac{x}{2}\right)+\left(\frac{x-1}{2}\right)$

vi) $(x)+\left(x+\frac{1}{n}\right)+\left(x+\frac{2}{n}\right)+....+\left(x+\frac{n-1}{n}\right)=(nx)+n-1,\ n\in N$

vii) $\left(\frac{n+1}{2}\right)+\left(\frac{n+2}{4}\right)+\left(\frac{n+4}{8}\right)+\left(\frac{n+8}{16}\right)+.... .....+\left(\frac{n+2^{n-1}}{2^n}\right)=2n-1, n\in N$

17. Fractional Part of x:

For $x\in R$, the function **f(x) = x - [x]** is called fractional part of x, it is denoted by **]x[** or **{x}**.

and is defined by $\{x\}=f$ if $x=n+f$ where $n\in I$ and $0\le f<1$

Eg : {2.7} = 0.7, {-3.6} = 0.4, {3} = 0

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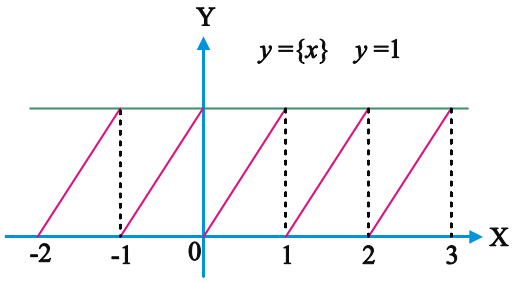
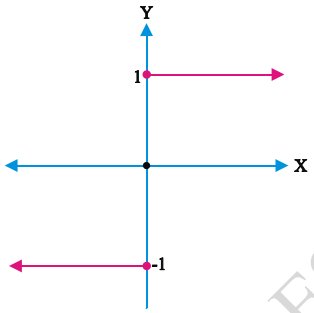


Note : Domain of $\{x\}$ is R and Range of $\{x\}$ is $[0,1)$

18. Signum Function: The signum function or signature function is defined as

$$Sgn(x)=\begin{Bmatrix}\frac{|x|}{x} & if & x\neq 0\\ 0 & if & x=0\end{Bmatrix}$$

$$i.e\ \ Sgn(x)=\begin{Bmatrix}1 & if & x>0\\ -1 & if & x<0\\ 0 & if & x=0\end{Bmatrix}$$



Note : The domain of $Sgn(x)$ is R and its

Range is $\{-1,0,1\}$

Properties of signum function

i) $\text{sgn}(Kx)=\text{sgn}(x);\ K\in N$

ii) $x.\text{sgn}(x)=|x|$

iii) $|x|.\text{sgn}(x)=x$

iv) $x.\text{sgn}(x).\text{sgn}(x)=x$

19. Exponential Function :–

If $a>0, x\in R$ then $f(x)=a^x$ is called exponential function

Domain of $a^x\ (a>0)$ is R

Range of $a^x\ (a>0)$ is $R^+=(0,\infty)$

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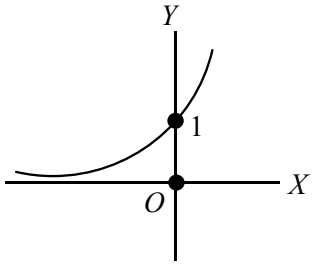
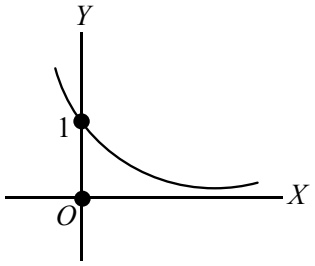
Graph of $a^x\ (a>1)$ **:**



Graph of $a^x\ (0<a<1)$



20. Logarithmic function :

Def : If 'N' and 'a' are any two positive real numbers and $a \neq 1$ and x is any real number such that $a^x = N$, then x is called logarithm of 'N' to the base 'a' i.e $x = \log_a N$

Note : i) $\log_a a = 1 (Q\ a = a^1)$

ii) $\log_a 1 = 0 (Q\ a^0 = 1)$

iii) $\log_a mn = \log_a m + \log_a n$

iv) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

v) $\log_x (m) + \log_x (n) - \log_x (p) - \log_x (q) = \log_x \left(\frac{mn}{pq}\right)$

vi) $\log_{b^n} a^m = \frac{m}{n} \log_b a$

vii) $\log_b a = \frac{1}{\log_a b}$

viii) $\log_b a . \log_a b = 1$

ix) $\log_b a \log_c b \log_d c = \log_d a$

x) $\log_b a = \frac{\log_x a}{\log_x b}$

xi) $a^{\log_a N} = N$

xii) $a^{\log_c b} = b^{\log_c a}$

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xiii) $\log_a x > 0 \Leftrightarrow x > 1, a > 1$ (or) $0 < x < 1, 0 < a < 1$

xiv) $\log_a x < 0 \Leftrightarrow x > 1, 0 < a < 1$ (or) $0 < x < 1, a > 1$

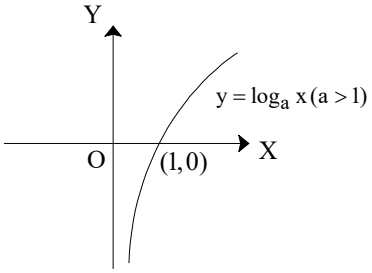
xv) $\log_a N < x \Leftrightarrow 0 < N < a^x, a > 1$

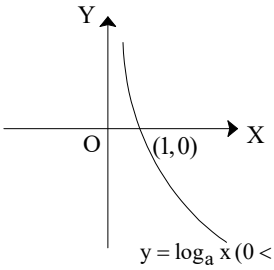
xvi) $\log_a N > x \Leftrightarrow N > a^x,\ 0 < a < 1$

xvii) $x < y \Leftrightarrow \log_a x < \log_a y,\ x > 0, y > 0, a > 1$

Definition : The function $f(x) = \log_a x$, where a is a positive constant and $a \neq 1$ is called **a logarithmic function.** Domain of f(x) **is** $(0, \infty)$ and its range is R

Graph of $\log_a x$





Common logarithm:- Logarithm to the base 10 is called common logarithm/briggs logarithm.

Natural logarithm:- logarithm to the base e is called natural logarithm/ Nepier logarithm, $\log_e x$ is denoted as lnx

21. **Explicit function :** A function $y = f(x)$ is said to be an explicit function of x if the dependent variable y can be expressed interms of the independent variable x.

E.g: i) $y = x - \cos x$,

ii) $y = x + \log_e x - 2x^3$.

22. **Implicit function:** A function $y = f(x)$ is said to be an implicit function of x if the dependent variable y can not be expressed in terms of the independent variable x only.

E.g:i) $3x^2 - 4x^2y^2 + 2xy = 7$

ii) $xy = \cos(x + y)$

23. **Equality of Functions:** The functions f and g are said to be equal if,

i) the domain of f = the domain of g

ii) $f(x) = g(x)$ for every x in domain.

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Ex : i) State whether $f(x)=x;\ g(x)=\sqrt{x^2}$ are identical or not

Sol. $D_f = D_g = R$ but $R_f = R$ where $R_g = [0,\infty) \Rightarrow f(x) \neq g(x)$

Ex : ii) State where $f(x)=\text{sgn}\left(x^2+1\right)$ and $g(x)=\sin^2 x+\cos^2 x$ are equal (or) not

Sol. $D_f = R = D_g$ and $f(x)=1\ \left(\because x^2+1>0\right)$

$$g(x)=1 \Rightarrow f(x)=g(x)$$

24. Composite function: If $f: A \to B$ and

$g: B \to C$ are functions then $gof: A \to C$ is defined by $(gof)(x) = g\{f(x)\}\ \forall x \in A$ is called the composite function of f & g

Note : If $f(x)=\dfrac{ax+b}{cx-a}$ then

i) $(fof)(x)=x$

ii) $(fofof)(x)=f(x)$

iii) $(fofofof.....)(x)=\begin{cases} x & \text{if f occurs in even number of times} \\ f(x) & \text{if f occurs in odd number of times} \end{cases}$

25. *Odd and Even functions :*

Odd function : A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for every real number x in the domain of f.

Even function : A function $f(x)$ is said to be even if $f(-x) = f(x)$ for every real number x in the domain of f.

Note:

i) The graph of an odd function is symmetric about origin and it is placed either in the first and third quadrant or in the second and fourth quadrant.

ii) The graph of an even function is symmetric about the y-axis.

iii) If a function $f(x)$ is such that $f(a-x) = f(a+x)$ then it is symmetric with respect to the line $x = a$.

iv) If a function $f(x)$ is such that $f(a-x) + f(a+x) = 0$ then it is symmetric with respect to the point $(a, 0)$.

v) If f and g are functions and gof is defined then

g	fog	gof
even	even	even
odd	odd	odd
odd	even	even
even	odd	even

vi) To express a given function $f(x)$ as the sum of an even and odd function, we write

$$f(x)=\frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]$$

where $\frac{1}{2}[f(x)+f(-x)]$ is an even function and $\frac{1}{2}[f(x)-f(-x)]$ is an odd function.

vii) If $f(x)$ is an odd function, then $f'(x)$ is an even function provided $f(x)$ is differentiable on R.

viii) If $f(x)$ is an even function, then $f'(x)$ is an odd function provided $f(x)$ is differentiable on R.

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ix) If (x, y) is point on the even function then $(-x, y)$ is also point on even function.

x) If (x, y) is a point on an odd function graphthen $(-x, -y)$ is also a point on the same odd function graph.

xi) A function may be neither even nor odd. for example $f(x) = \sin x + x^2$ is neither even nor odd.

xii) $f(x) = 0$ is the function which is both even and odd.

xiii) $f(x) = c$ is an even function

26. Extension of a function:

Let a function $f(x)$ defined on the interval $[0, a]$, it can be extended on $[-a, a]$ sothat $f(x)$ is either even or odd function on $[-a, a]$

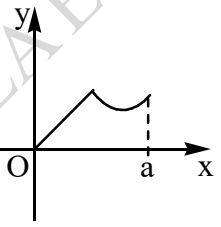
i) Even Extension

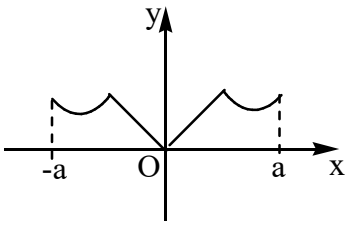
As $f(x)$ is defined on $[0, a]$

$f(-x)$ is defined on $[-a, 0]$, then even extension i.e, $F_e(x)$ is defined as

$F_e(x) = f(x);\ x \in [0, a] = f(-x);\ x \in [-a, 0]$

Graphically : If the graph of $y = f(x), x \in [0, a]$ is as follows then even functions is given as

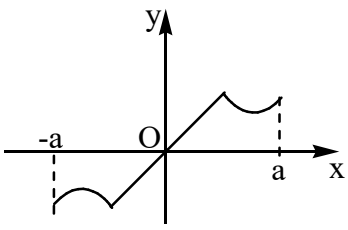




i.e, Even extension function of $y = f(x)$ is obtained as a function which is symmetrical about y-axis.

ii) Odd Extension : As $f(x)$ is defined on $[0, a]$, then odd extension i.e., $F_0(x)$ is given as

$F_0(x) = f(x);\ x \in [0, a] = -f(-x);\ x \in [-a, 0]$



i.e, odd extension is obtained as a function which is symmetrical about the origin.

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27. **Periodic function:** A function f(x) is said to be periodic function if there exists a positive real number T, such that $f(x + T) = f(x),\ \forall x \in D_f$, where the least positive real number T is called its fundamental period.

28. **Algebraic functions:** A function f is said to be an algebraic function if it arises due to a finite number of fundamental operations like addition, subtraction, multiplication, division and root extraction etc, on polynomial functions

29. **Transcendental function:** The functions which are not algebraic are called transcendental functions.

E.g: i) Exponential functions

ii) Logarithmic functions

iii) Trigonometric functions

iv) inverse trigonometric functions

E.g: $f(x) = x^3 + 3x^2 - \sqrt{x^2 - x} + \sqrt[3]{x}$ is an

algebraic function where as $g(x) = x + \sin^{-1} x.$

$h(x) = e^x.\cosh^{-1} x$ are transcendental functions

30. **Algebra of real valued functions :**

Definition : Let $f : A \to B$ is a function and

i) If $A \subseteq R$ then f is called a real variable function.

ii) If $B \subseteq R$ then f is called a real valued function.

iii) If $A \subseteq R, B \subseteq R$ then f is called a real function.

Properties: If f and g are real valued

functions with domain A and B respectively, then both f and g are defined on $A \cap B$ when $A \cap B \neq \phi$.

Let $f : A \to R$ & $g : B \to R$ then

i) $(f \pm g)(x) = f(x) \pm g(x), \forall x \in A \cap B$

ii) $(fg)(x) = f(x).g(x), \forall x \in A \cap B$

iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in A \cap B \ \&\ g(x) \neq 0$

iv) $(f \pm k)(x) = f(x) \pm k, k \in R, x \in A$

v) $(kf)(x) = kf(x), k \in R, x \in A$

vi) $f^n(x) = \{f(x)\}^n, n > 0, x \in A$

vii) $|f|(x) = |f(x)|, x \in A$

31. ***Some important functional equations***

If x, y are independent variables then

i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \log_e x$ or $f(x) = 0$

ii) $f(xy) = f(x) \,.\, f(y) \Rightarrow f(x) = x^n,\ x \in R.$

iii) $f(x + y) = f(x) \,.\, f(y) \Rightarrow f(x) = a^{kx}.$

iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx.$

v) $f(x + y) = f(x) = f(y) \Rightarrow f(x) = k,$ where k is constant.

vi) By considering a general n^{th} degree polynomial and writting the expression,

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$f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right) \Rightarrow f(x)=1\pm x^n$

vii) If f(x) is a function such that $f(x+y)+f(x-y)=2f(x)f(y)$ then $f(x)=\frac{k^x+k^{-x}}{2}$ or $f(x)=\cos x$

viii) A continuous function $f(x)$ takes only rational values (irrational values) for all $x \Rightarrow f(x)$ is a constant function.

32. One-one function (Injection) :

A function $f: A \to B$ is one-one (injection) if distinct elements of A have distinct f images in B.

33. ***Many - one function :*** A function $f: A \to B$ is said to be many-one if there exists atleast two distinct elements in A whose images are same.

(or) A function which is not one – one is called many – one function

Note :

i) The condition for a function $f: A \to B$ to be a one – one function is $n(A) \le n(B)$

ii) If A and B are any two finite sets having m and n elements respectively, then the number of one-one functions from A to B would be

$$=\begin{Bmatrix} {}^nP_m, \text{ if } n \ge m \\ 0 \text{ , if } n < m \end{Bmatrix}$$

ii) If $n(A) \le n(B)$ then the number of many to one functions from A to B is $[n(B)]^{n(A)} - {}^{n(B)}P_{n(A)}$

34. ***Working Rule for checking whether the function $f: A \to B$ is one-one (or) many-one:***

i) Let $x_1, x_2 \in$ domain of f. If $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$ for every x_1, x_2 in the domain, then f is one-one, other wise many-one.

ii) Consider any two points $x_1, x_2 \in A$. Take $f(x_1) = f(x_2)$ and solve the equation if we get $x_1 = x_2$ only then f is one-one

iii) If the elements in the function is given in order pairs then the elements in second place must be different then given function is one – one

iv) **Horizontal line test :** If any straight line parallel to x-axis intersects the graph of the function atmost at one point then the function is one-one, other wise it is many- one. (i.e., intersects the graph of the function in atleast two points).

v) All even functions are many-one

vi) If the function is continuous and entirely strictly increasing or strictly decreasing in the domain, then f is one - one, other wise many-one .

vii) All periodic functions are many one

viii) $f: R \to R, f(x) = ax + b (a \ne 0)$ is one – one

ix) $f: R \to R, f(x) = ax^2 + bx + c (a \ne 0)$ is not one – one

35. Onto function (Surjection) :

$f: A \to B$ is called an onto or a surjection if every element of B has atleast one pre-image in A.

(or)

If $f: A \to B$ is onto (a surjection) then the range = codomain of 'f' i.e., $f(A) = B$.

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Note : i) The condition for a function $f : A \to B$ to be a surjection from A to B is $n(A) \geq n(B)$.

ii) If n(A) = n(≥ 2), n(B) = 2 then the number of onto functions from A to $B = 2^n - 2$

iii) If $n(A) = r$ and $n(B) = n(\geq 2)$. Then the number of surjections from A to B is

$$n^r - {}^nC_1(n-1)^r + {}^nC_2(n-2)^r - {}^nC_3(n-3)^r + \ldots + (-1)^{n-1}.{}^nC_{n-1}.$$

iv) If $n(A) < n(B)$ then the no. of onto functions is zero.

36. Into function : If $f : A \to B$ is not onto then it is called an into function. i.e., there is no pre- image for atleast one element of B in A then *f* is into function.

37. Working Rule for checking whether the function $f : A \to B$ is onto or into:

i) Let $y \in B$ and $y = f(x)$ and from this find x in terms of y. $\forall\ y \in B$ if there exists atleast one $x \in A$, then f is onto. Otherwise into. i.e., if range of $f = B$, then f is onto, other wise it is into.

ii) Horizontal line test : If every straight line parallel to x-axis from points in the codomain intersects the graph of the function at atleast one point then f is onto.

If the line does not cut the graph of $y = f(x)$ then f is into

38. Bijection (or) one-one & onto function:

If $f : A \to B$ is both an injection and a surjection then f is said to be bijection or one to one and onto from A to B.

Note : i) If A,B are finite sets and $f : A \to B$ is a bijection then $n(A) = n(B)$.

ii) Identity function on any non empty set A is bijection.

iii) If A,B are finite sets and $n(A) = n(B)$ then number of bijective functions defined from A to B is $(n(A))!$

iv) A constant function is bijection if n(A)= n(B)=1

v) Horizontal line test: If every straight line parallel to x-axis from the points in codomain inter sects the graph of the function at only one point then f is bijection.

39. *Inverse Function :* If the function

$f : A \to B$ is both one-one and onto, then we define inverse function $f^{-1} : B \to A$ by the rule

$y = f(x) \Leftrightarrow f^{-1}(y) = x,\ \forall x \in A,\ \forall y \in B$

Note : i) If f is invertible then f must be bijective i.e one-one and onto.

ii) If the inverse of a function exists then it is said to be invertible. The inverse of a function if exists then it is unique

iii) Graph of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about $y = x$ and intersects on the line $y = x$, and also $f(x) = f^{-1}(x) = x$ when ever graph intersects

40. Working Rule to find the inverse of a function

Let $f : X \to Y$ be a bijection function.

put $f(x) = y$. Solve the equation $y = f(x)$ to obtain x in terms of y. Interchange x and y to obtain the inverse of f

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41. Properties of composite function :

i) If $f: A \to B, g: B \to C$ are one-one then $gof: A \to C$ is also one-one.

ii) If $f: A \to B, g: B \to C$ are onto then $gof: A \to C$ is also onto.

iii) If $gof: A \to C$ is one-one then f is one-one.

iv) If $gof: A \to C$ is onto then g is onto.

v) If $f: A \to B$ is a function then $foI_A = I_B of = f$.

vi) If $f: A \to B$ and $g: B \to C$ are bijections then $gof: A \to C$ is a bijection & $(gof)^{-1} = f^{-1}og^{-1}$.

vii) If $f: A \to B$ is a bijection then $fof^{-1} = I_B, f^{-1}of = I_A$.

viii) If $f: A \to B$ and $g: B \to A$ are functions such that $g\,of = I_A$ and $f\,og = I_B$ then $g = f^{-1}$.

ix) If $f: A \to A$ is a bijection then $f\,of^{-1} = f^{-1}of = I_A$.

x) In general $f\,og \neq g\,of$.

xi) If $f: A \to B$, $g: B \to C$ and $h: C \to D$ are functions then $ho(gof) = (hog)\,of$.

42. *Working Rules for finding the domain of a function :*

Algebraic Functions :

i) Denominator should be non-zero.

ii) Expression under the even root should be non-negative i.e. ^{3} 0

Trigonometric Functions :

i) $\sin x$ and $\cos x$ are defined for all real values of x.

ii) $\tan x$ and $\sec x$ are defined for all real values of x except $x = (2n+1)\frac{\pi}{2}$, where $n \in I$.

iii) $\cot x$ and $\operatorname{cosec} x$ are defined for all real values of x except $x = n\pi$, where $n \in I$.

Inverse trigonometric Functions :

i) $\sin^{-1} x$ and $\cos^{-1} x$ are defined for $-1 \le x \le 1$.

ii) $\tan^{-1} x$ and $\cot^{-1} x$ are defined for $x \in R$

iii) $\sec^{-1} x$ and $\operatorname{csc}^{-1} x$ are defined for $x \le -1$ or $x \ge 1$

43. *Working Rules for finding the range of a function y = f(x)*

The set of values of 'x' for which $f(x)$ is defined is called the domain of $f(x)$ and is denoted by D_f.

The set of images of the elements of the domain (out comes) is called the range of $f(x)$ and is denoted by R_f.

Study of range is very important. There are no fixed methods to find range. Some of the methods to find the range are given as follows.

Type –1 : If the domain is a set containing finite number of elements then find the corresponding images under the given function and the set of all images so obtained is the range.

Type –2 : To find the range of a function whose domain is $(-\infty, \infty)$, we proceed as follows. Let $y = f(x)$

Now express 'x' interms of 'y' i.e, $x = g(y)$ (if possible)

Now find 'y' for which 'x' is defined.

Type – 3 : If the graph of the function is known, then the range of the function can be obtained easily from the graph.

Type – 4 : Usage of $A.M \ge G.M \ge H.M$ on the given positive quantities

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Note : Range of $x+\frac{1}{x}$ is $(-\infty,-2]\cup[2,\infty)$

Type – 5 : The range of $y=ax^2+bx+c$ is $\left[\frac{4ac-b^2}{4a},\infty\right)$ (or) $\left(-\infty,\frac{4ac-b^2}{4a}\right]$ according as a > 0 or a < 0. If the domain of ax^2+bx+c is restricted then draw its graph and decide its range

Type – 6 : To find the range of $f(x)=\frac{P(x)}{Q(x)}$, where degree is '2' then $yQ(x)=P(x)\Rightarrow$ we have a quad. equation in 'x'

$\Delta\geq 0$ ($\because$ 'x' is real) $\Rightarrow$ range will come

Type – 7 : If $f(x)$ is strictly increasing function in $[a,b]$ then its range $=[f(a),f(b)]$

If $f(x)$ is strictly decreasing in $[a,b]$ then its range $=[f(a),f(b)]$

Type – 8 : To find the range $f(x)$ in $[a,b]$, find $f'(x)$ and put $f'(x)=0\Rightarrow x=\alpha,\beta,\gamma,.....$ accept these values if they line in $[a,b]$ find these images and also find $f(a),f(b)$. The least, greatest values thus obtained are called Max, Min. values of $f(x)$ in $[a,b]$

Type – 9 : To find range of $f(x)$ in (a,b) follow the above procedure is $f'(x)=0\Rightarrow x=\alpha,\beta,\gamma,.....$ If m, M are the least, greatest values.

If $\underset{x\to a^+}{Lt}(f(x))$ and $\underset{x\to b^-}{Lt}(f(x))$ both are greater than 'm' then 'm' is least value. Similarly, if $\underset{x\to a^+}{Lt}(f(x))$ and $\underset{x\to b^-}{Lt}(f(x))$ both are less than 'M', then M is the greatest value. Otherwise we cannot decide the least, greatest values.

Type –10 : If various functions of different natures are present in the given problem, then find the common domain and have the range.

Note : i) The range of

$|x-a|+|x-b|$ $(b>a)$ ***is*** [b- a,¥).

ii) **The range of** acosx+bsinx+c ***is*** $\left[c-\sqrt{a^2+b^2},c+\sqrt{a^2+b^2}\right]$

iii) The range of $\sin^3 x+\cos^3 x$ is R

iv) If $a^2+b^2+c^2=k$, then range of $ab+bc+ca$ is $\left[-\frac{k}{2},k\right]$

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44. THE DOMAINS AND THE RANGES OF SOME STANDARD FUNCTIONS

Algebraic Functions

Function	Domain	Range
$\log_a x$	$(0,\infty)$	R
a^x	R	$(0,\infty)$
$[x]$	R	Z
$\{x\} = x - [x]$	R	$[0,1)$
$\lvert x \rvert$	R	$[0,\infty)$
$\sqrt{x}$	$[0,\infty)$	$[0,\infty)$
$x + \frac{1}{x}$	$R - \{0\}$	$(-\infty,-2] \cup [2,\infty)$

Trigonometric Functions

Function	Domain	Range
$\sin x$	R	$[-1,1]$
$\cos x$	R	$[-1,1]$
$\tan x$	$R - \left\{(2n+1)\frac{\pi}{2} / n \in Z\right\}$	R
$\cot x$	$R - \{n\pi / n \in Z\}$	R
$\sec x$	$R - \left\{(2n+1)\frac{\pi}{2} / n \in Z\right\}$	$(-\infty,-1] \cup [1,\infty)$
$\operatorname{cosec} x$	$R - \{n\pi / n \in Z\}$	$(-\infty,-1] \cup [1,\infty)$

Inverse Trigonometric Functions

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Function	Domain	Range
$\sin^{-1} x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0,\pi)$
$\sec^{-1} x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\{\pi/2\}$
$\operatorname{cosec}^{-1} x$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Hyperbolic Functions

$$\sinh x=\frac{e^x-e^{-x}}{2},\ \cosh x=\frac{e^x+e^{-x}}{2}$$

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}},\ \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}\ \sec hx=\frac{2}{e^x+e^{-x}},\ \operatorname{cosec} hx=\frac{2}{e^x-e^{-x}}$$

Function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1,\infty)$
$\tanh x$	R	$(-1,1)$
$\coth x$	$R-\{0\}$	$(-\infty,-1)\cup(1,\infty)$
$\operatorname{sech} x$	R	$(0,1]$
$\operatorname{cosec} h x$	$R-\{0\}$	$R-\{0\}$

Inverse Hyperbolic Functions

$$\sinh^{-1} x=\log_e\left(x+\sqrt{x^2+1}\right),\forall x\in R$$

$$\cosh^{-1} x=\log_e\left(x+\sqrt{x^2-1}\right),\forall x\in[1,\infty)$$

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$$\tan h^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \forall x \in (-1,1)$$

$$\coth^{-1}x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right), \forall x \in (-\infty,-1)\cup(1,\infty)$$

$$\sec h^{-1}x = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right), \forall x \in (0,1]$$

$$\cos ec\, h^{-1}x = \log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right), \forall x > 0$$

$$= \log\left(\frac{1-\sqrt{1+x^2}}{x}\right), \forall x < 0$$

Function	Domain	Range
$\sinh^{-1}x$	R	R
$\cosh^{-1}x$	$[1,\infty)$	$[0,\infty)$
$\tanh^{-1}x$	$(-1,1)$	R
$\coth^{-1}x$	$(-\infty,-1)\cup(1,\infty)$	$R-\{0\}$
$\text{sech}^{-1}x$	$(0,1]$	$[0,\infty)$
$\cos ec\, h^{-1}x$	$R-\{0\}$	$R-\{0\}$

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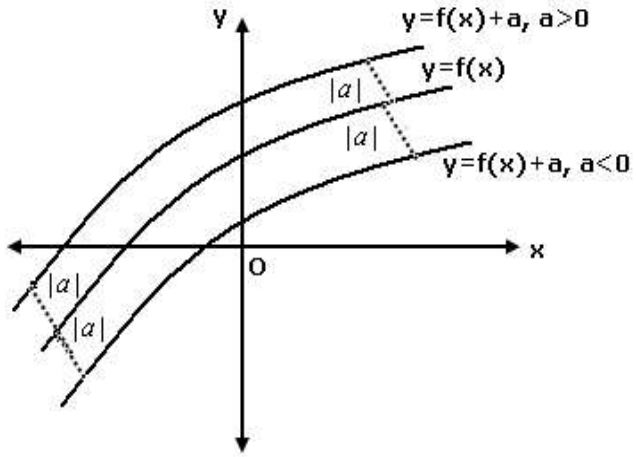
45. GRAPHICAL TRANSFORMATIONS :

With the help of the given basic function (curve) $y = f(x)$, drawing the other graphs by allowing some changes in x (independent variable) and in y (dependent variable) is called trasformation of graphs. Now let us observe some important transformations of graphs.

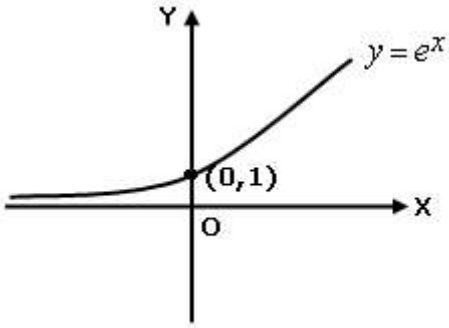
i) Drawing the graph of $y = f(x)+a, (a \in R)$

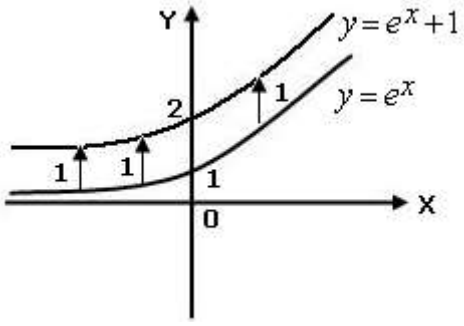
from the known graph of $y = f(x)$**.**

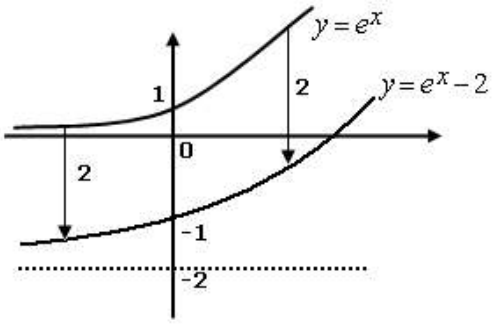
If $a > 0$ then shift the graph of $f(x)$ in upward direction through a distance 'a' (or) shift the X-axis downward through a distance 'a' and if a<0 then shift the graph of $f(x)$ in downward direction through a distance 'a' or shift the X-axis upward through a distance 'a' units.



Ex1 : Draw the graph of $y = e^x + 1,\ y = e^x - 2$ with the help of $y = e^x$







Ex2 : plot $y = |x| + 2$ and $y = |x| - 2$ with the help of $y = |x|$

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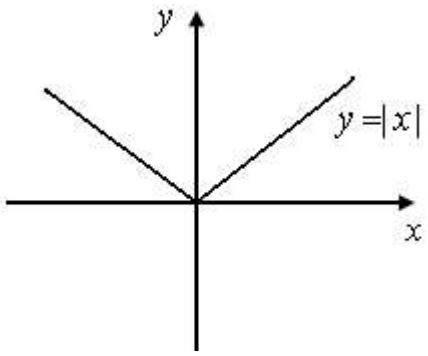
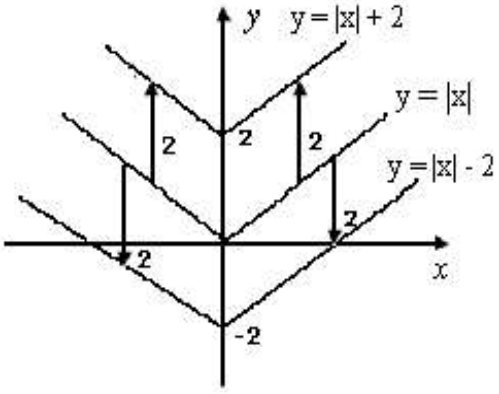
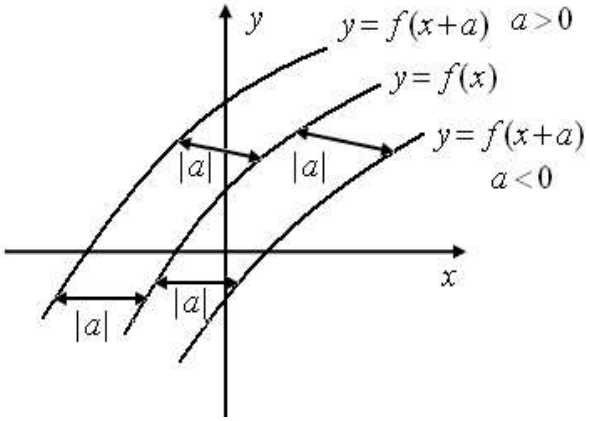


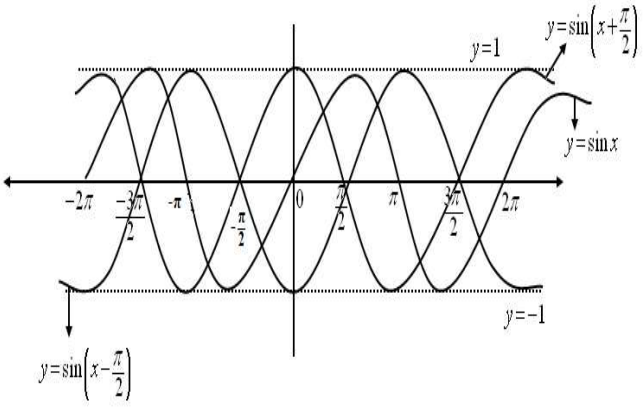


ii) **Drawing the graph of $y = f(x+a)\,(a \in R)$ with the help of the graph $y = f(x)$**

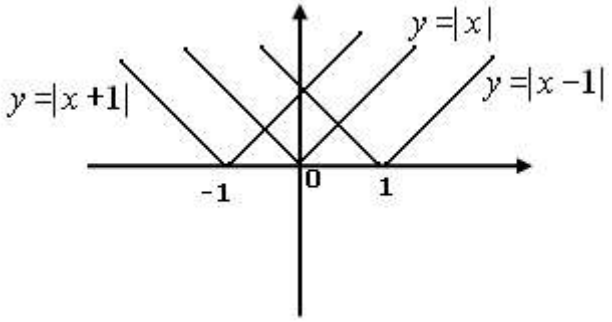
If a>0, then shift the graph of $f(x)$ in the left hand direction through a distance 'a' (or) shift the y-axis to the right hand direction by distance 'a'. Similarly if a<0 then shift the graph of $f(x)$ in the right hand direction (or) shift the y-axis to the left by the distance 'a' units.



Ex1 : Plot the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ and $y = \sin\left(x + \frac{\pi}{2}\right)$ with the help of $y = \sin x$

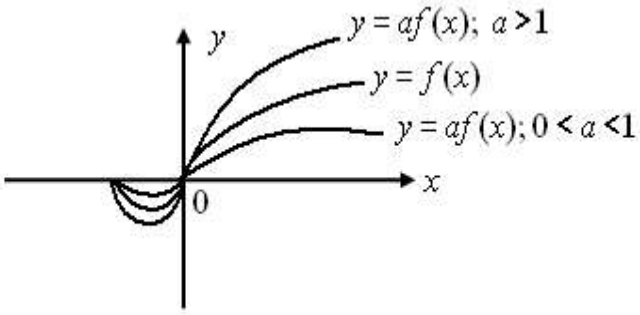


Ex2 : Draw the graph of $y = |x-1|$ and $y = |x+1|$ with the help of $y = |x|$



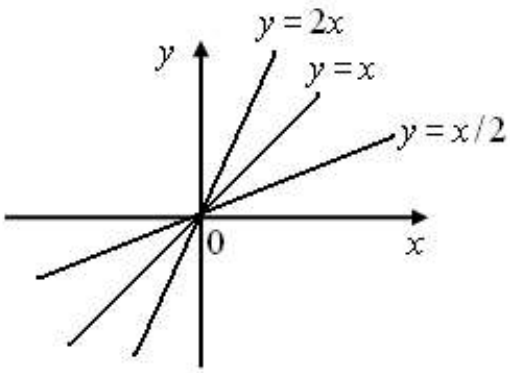
iii) Drawing the graph of $y = af(x)$ from the known graph $y = f(x)$

The graph of the function $y = af(x)$ $(a > 0)$ is obtained from the graph of $y = f(x)$ by "stretching" vertically 'a' time (a>1) and compressing vertically $\frac{1}{a}$ times (0 < a < 1) along the y - axis.

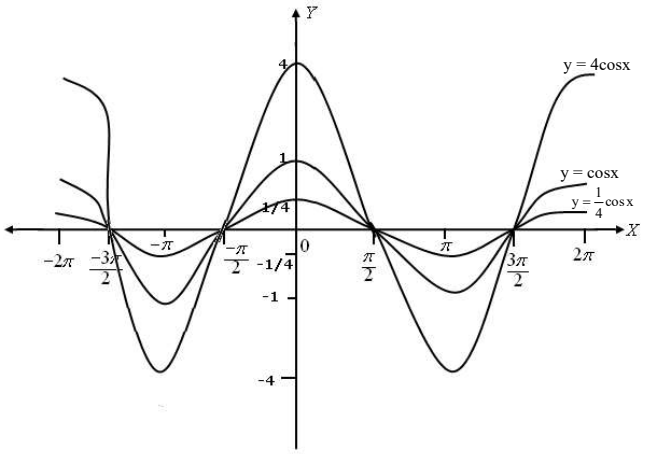


Ex1 : Plot $y = 2x$ and $y = \frac{x}{2}$ with the help of $y = x$

i.e., $y = 2x$ is the stretch of $f(x)$ two times along the y-axis and $y = \frac{1}{2}x$ is the shrink of $f(x)$ two times along the y-axis.

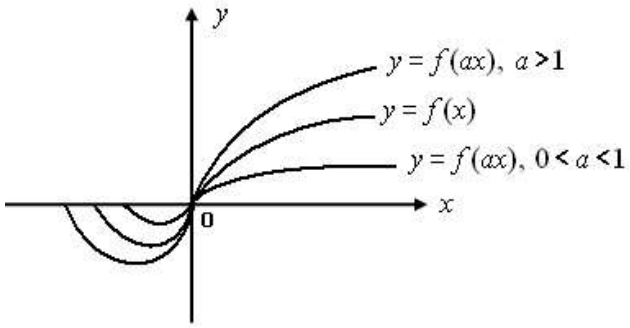


Ex2 : Draw the graph of $y = 4\cos x$ and $y = \frac{1}{4}.\cos x$ with the help of $y = \cos x$

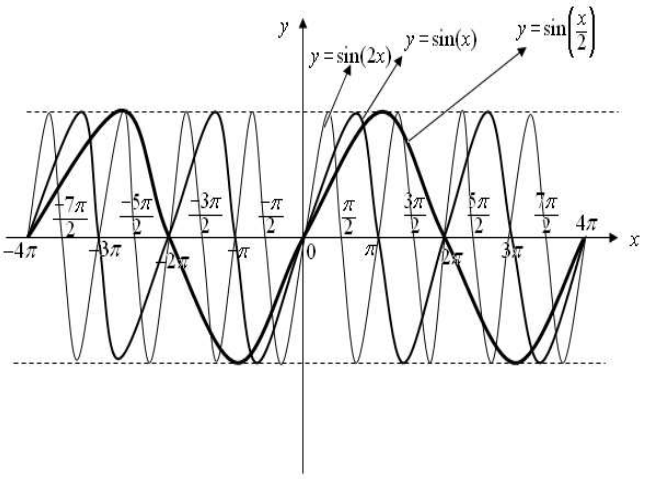


iv) Drawing the graph of $y = f(ax)$ **from the graph of** $y = f(x)$

The graph of $y = f(ax)$ $(a > 0)$ is obtained from $y = f(x)$ by stretching horizontally (along x-axis) from the y-axis $\frac{1}{a}$ times $(0 < a < 1)$ and compressing horizontally from the y-axis 'a' times $(a > 1)$

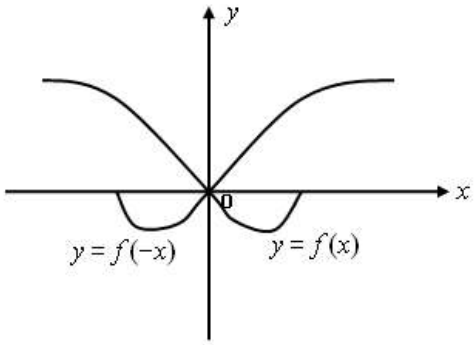


Ex : Draw the graph of $y = \sin(2x)$ and $y = \sin\left(\frac{x}{2}\right)$ from the graph of $y = \sin x$.

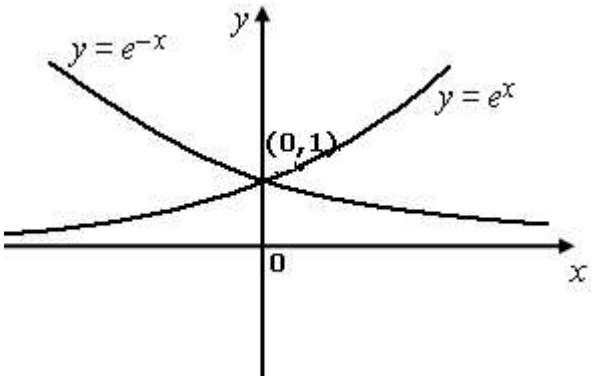


v) Drawing the graph of $y = f(-x)$ from the known graph of $y = f(x)$

The graph of $y = f(-x)$ is obtained from the graph of the function $y = f(x)$, $y = f(-x)$ is symmetrical to that of the function $y = f(x)$ about the y-axis. ie., take reflection of $f(x)$ considering y-axis as plane mirror i.e., turn the graph of $f(x)$ by 180^0 about y-axis.



Ex : Draw the graph of $y = e^{-x}$ with the help of $y = e^{x}$

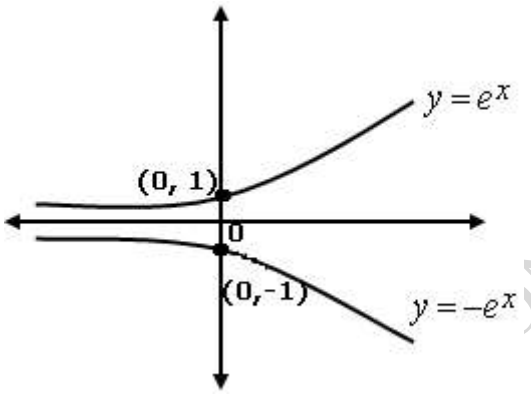


i.e., $y = e^{-x}$ is the image of $y = e^{x}$ w.r.t y-axis.

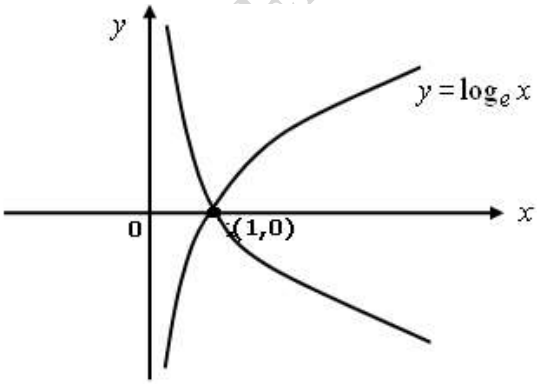
vi) Drawing the graph of $y=-f(x)$ from the known graph of $y=f(x)$

The graph of $y=-f(x)$ is the image of $y=f(x)$ in the X-axis as plane mirror (OR) to get the graph of $y=-f(x)$, turn the graph of $f(x)$ by 180^0 about the X-axis.

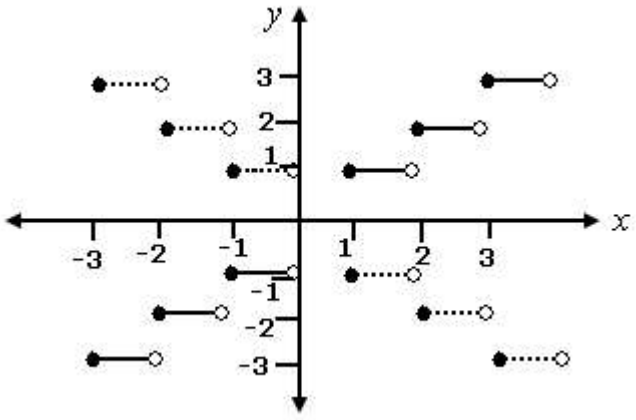
Ex. 1) Plot the curve $y=-e^x$ with the help of $y=e^x$



Ex1 : Plot the curve $y=-\log_e x$ with the help of $y=\log_e x$



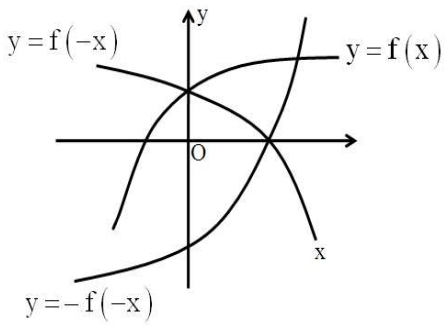
Ex2 : Draw the graph of $y=-[x]$ (G.I.F) with the help of $y=[x]$ (G.I.F)



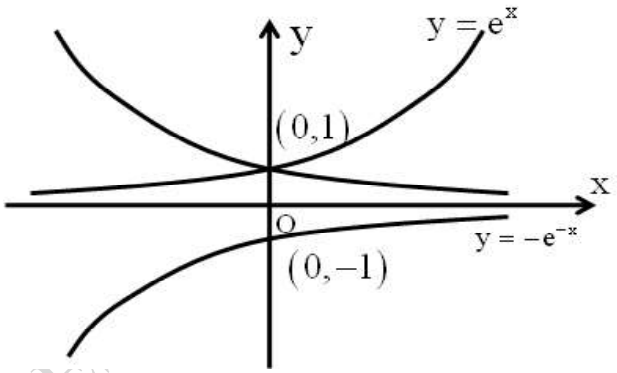
vii) Drawing the grapgh of $y=-f(-x)$ from the known graph of $y=f(x)$

Since $f(-x)$ is symmetrical to that of $f(x)$ about the y-axis and $-f(-x)$ is symmetrical to that of $f(-x)$ about the x-axis.

i.e., first image of $f(x)$ about y-axis later its image about the x-axis

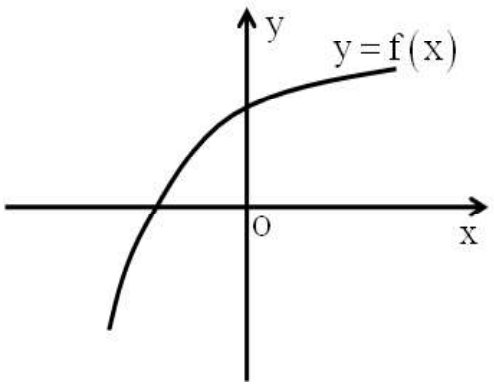


Ex : Drawn the graph of $y=-e^{-x}$ with the help of the known graph $y=e^{x}$



viii) Drawing the graph of $y=|f(x)|$ from the known graph of $y=f(x)$

The graph of $y=|f(x)|$ is obtained from the graph of $y=f(x)$, the portion lying above the x-axis remains unchanged. Now take the image of negative part of $f(x)$ (i.e., the portion below the x-axis) in the x-axis as plane mirror.



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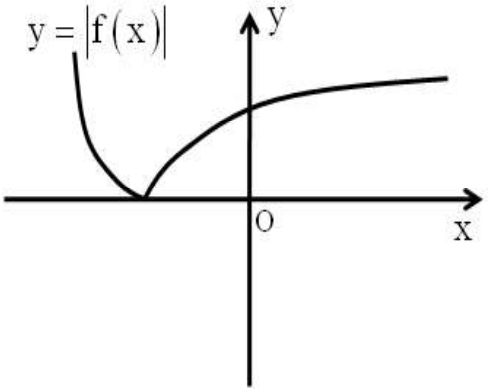
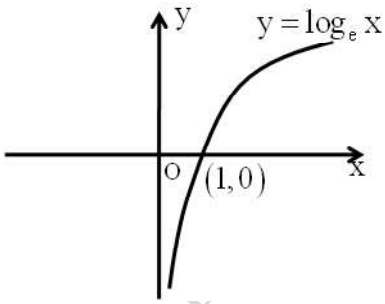
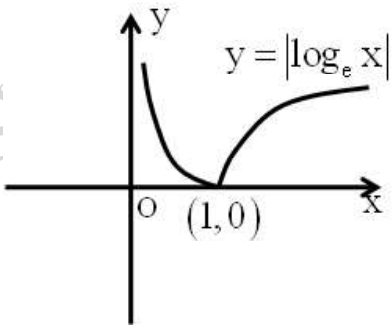
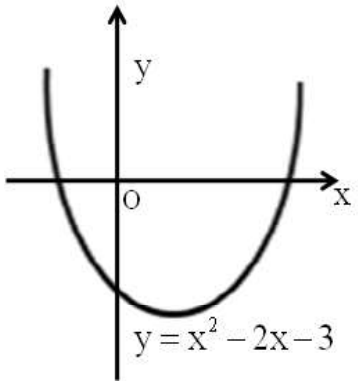


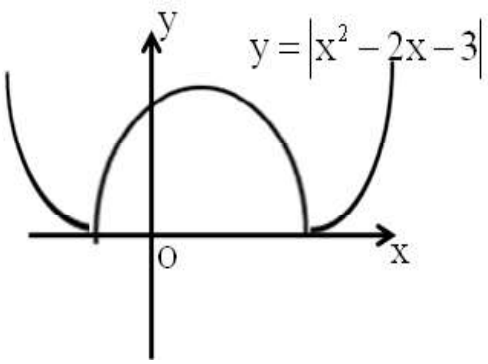
Ex1 : Draw the graph of $y = |\log_e x|$ with the help of the known function $y = \log_e x$.





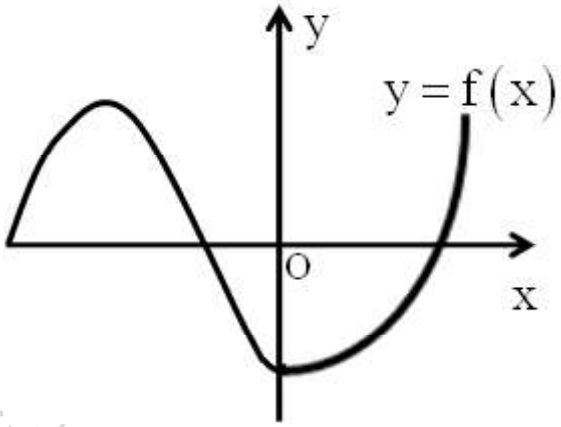
Ex2 : Draw the graph of $y = |x^2 - 2x - 3|$ with the help of $y = x^2 - 2x - 3$

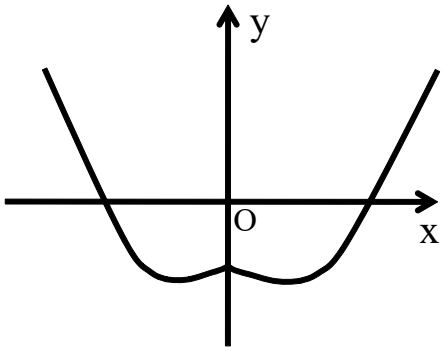




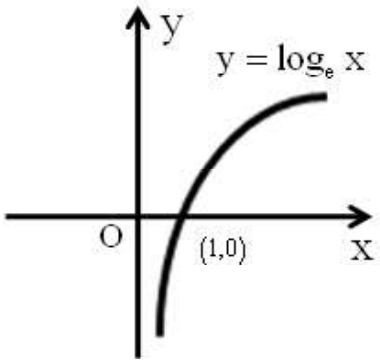
ix) Drawing the graph of $y = f(|x|)$ from the known graph of $y = f(x)$

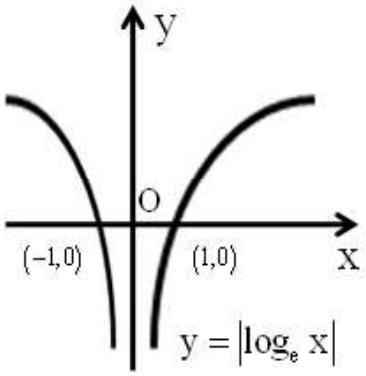
First draw the graph of $y = f(x)$ leave the graph lying right side of the y-axis as it is and also take its image w.r.t y-axis the portion of $y = f(x)$ in the left of y-axis is ignored.





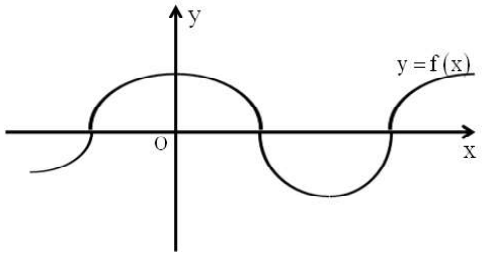
Ex : Sketch the graph of $y = \log_e |x|$ with the help of $y = \log_e x$.

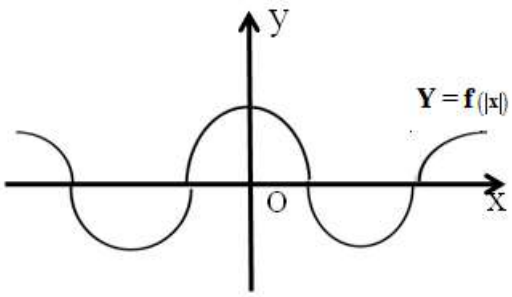




x) Drawing the graph of $y=\left|f\left(|x|\right)\right|$ from the known graph of $y=f\left(x\right)$

Here first we drawn the graph of $y=f\left(|x|\right)$ from the previous method, later the part above the x-axis remains unchanged and the part below the x-axis will be shifted to the above the x-axis (i.e., take its image w.r.t x-axis)





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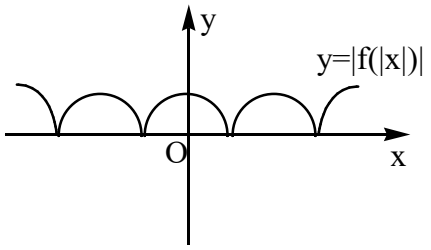
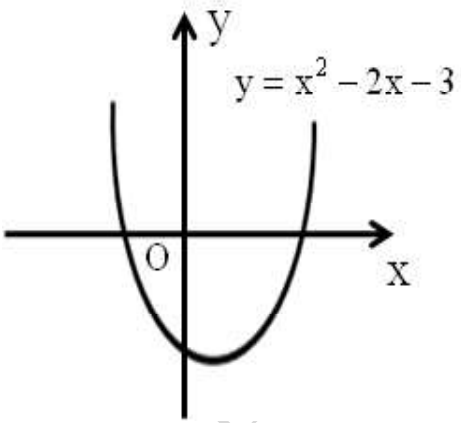
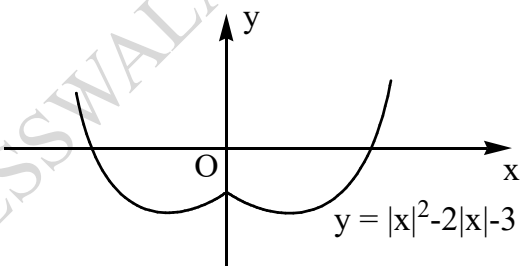
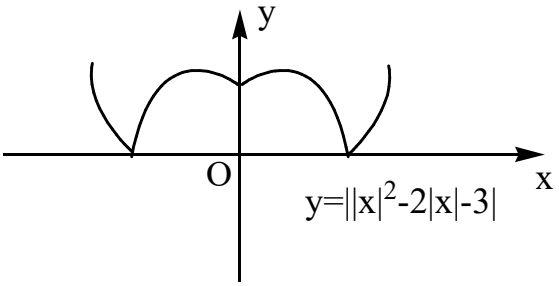


Ex : Sketch the graph of $y = \left| |x|^2 - 2|x| - 3 \right|$ with the help of $y = x^2 - 2x - 3$







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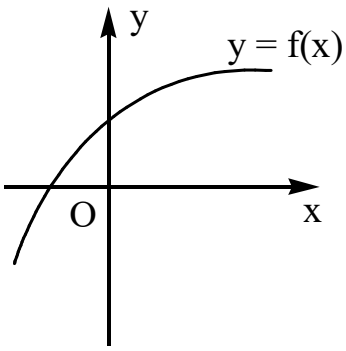
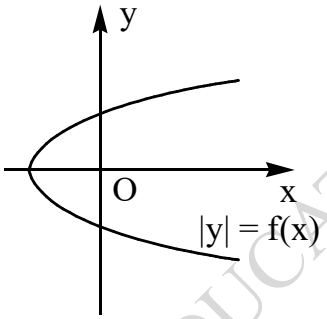
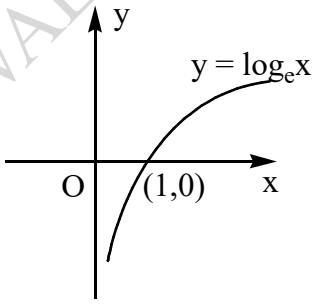
xi) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

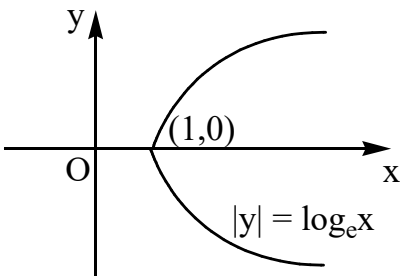
The graph of $|y| = f(x)$ is obtained from $y = f(x)$, the portion below the x-axis is ignored, the portion above the x-axis will be same and also have to take its image w.r.t x-axis (Here $f(x) < 0$ is not possible)





Ex : Draw the graph of $|y| = \log_e x$ with the help of $y = \log_e x$





xii) Drawing the graph of $|y| = |f(x)|$ from the known graph of $y = f(x)$

First we draw $|y| = f(x)$ from the previous method, then the graph of $|y| = |f(x)|$ is the portion of graph $|y| = f(x)$ lying above the x-axis remains unchanged. Its other portion located below the x-axis then take its image w.r.t x-axis

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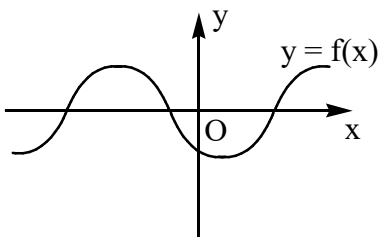
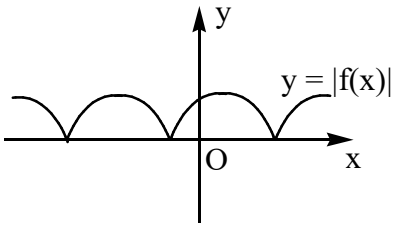
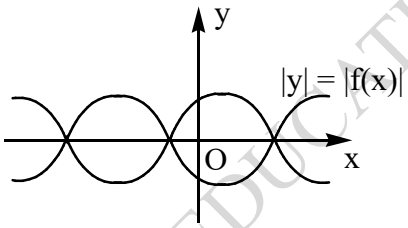
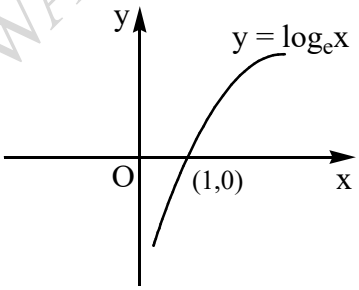
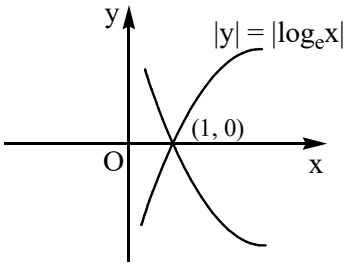






Ex : Draw the graph of $|y| = |\log_e x|$ from the graph of $y = \log_e x$





xiii) Drawing the graph of $|y| = |f(|x|)|$ from the known graph of $y = f(x)$

The graph of $|y| = f(|x|)$ is obtained from the graph of $y = f(x)$. First draw the graph of $y = |f(|x|)|$ from the previous technique take its image w.r.t x-axis to get the graph of $|y| = |f(|x|)|$

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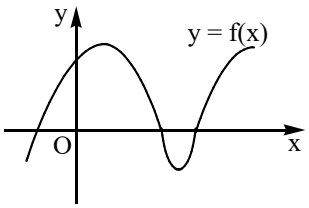
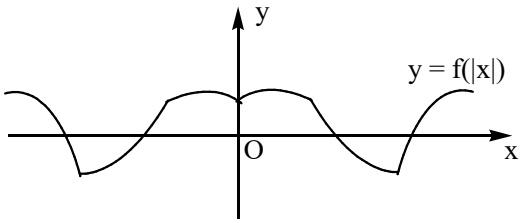
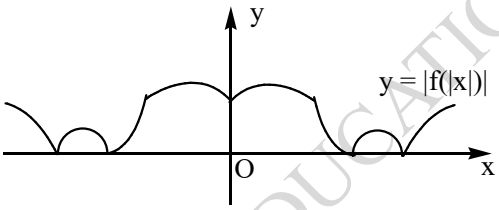
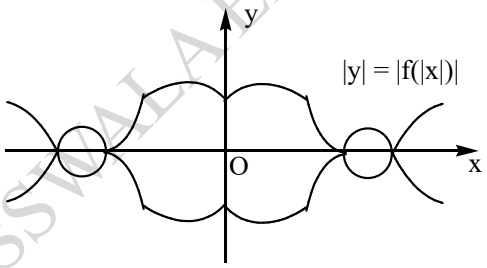
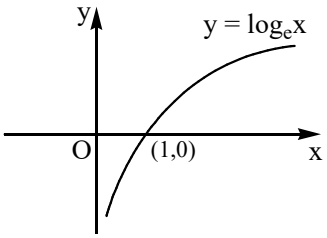


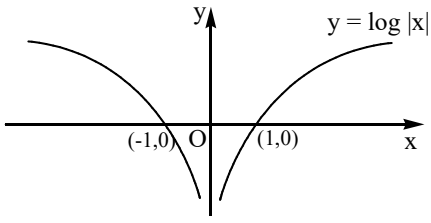






Ex : Draw the graph of $|y| = |\log|x||$ with the help of $y = \log x$





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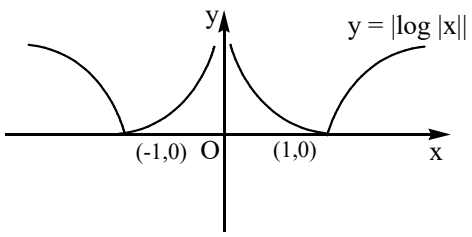
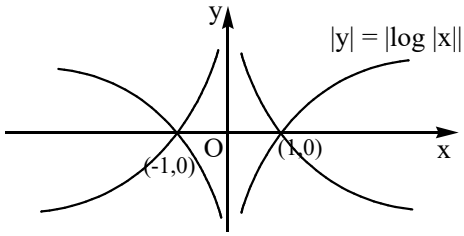
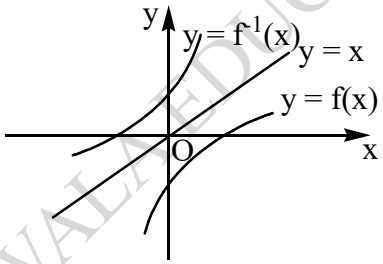




xiv) Drawing the graph of $y = f^{-1}(x)$ from the known graph of $y = f(x)$

The graph of $y = f^{-1}(x)$ is obtained by taking the reflection of $y = f(x)$ w.r.t the mirror line $y = x$



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