4 DETERMINANTS

SYNOPSIS

1. Determinant of a square matrix :

The determinant of the square matrix $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is the unique number $a_1b_2 - a_2b_1$ and is denoted by

$$\text{Det } A = |A| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

2. Minor of an element in a square matrix:

Let $A=\left[a_{ij}\right]$ be a square matrix. The minor of an element a_{ij} in A is the determinant of the square matrix that remains after deleting the i^{th} row and j^{th} column of A. It is denoted by M_{ij}.

3. Cofactor of an element of a square

matrix : The cofactor of an element in the i^{th} row and the j^{th} column of a matrix is defined as its minor multiplied by $(-1)^{i+j}$. If A_{ij} is the cofactor of a_{ij}, then $A_{ij} = (-1)^{i+j} M_{ij}$

$$\text{Let A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

i) Minor of a_{23} is $(M_{23}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

ii) Cofactor of a_{31} is $A_{31} = (-1)^{3+1} M_{31} = + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

iii) The cofactors of a_{11}, a_{12}, a_{13}..... are denoted by A_{11}, A_{12}, A_{13},.....

iv) For the matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, the cofactor matrix is given by $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

Eg : $B_2 = (-1)^{2+2} \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = (a_1c_3 - a_3c_1)$

4. DETERMINANT VALUE OF A SQUARE MATRIX

Definition : The determinant of a square matrix is the sum of the products of the elements of any row or column with their corresponding cofactors.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\Rightarrow |A| = a_1A_1 + b_1B_1 + c_1C_1$$

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$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$$

Clearly

$$|A| = a_1A_1 + b_1B_1 + c_1C_1$$

$$= a_2A_2 + b_2B_2 + c_2C_2$$

$$= a_3A_3 + b_3B_3 + c_3C_3$$

$$= a_1A_1 + a_2A_2 + a_3A_3$$

$$= b_1B_1 + b_2B_2 + b_3B_3$$

$$= c_1C_1 + c_2C_2 + c_3C_3$$

5. Properties of determinants:

i) The determinant of a square matrix changes its sign when any two rows (or columns) are inter changed

ii) If two rows (columns) of a square matrix are identical or proportional then the value of the determi nant is zero.

iii) If all the elements of a row (column) of a square matrix are multiplied by a number K then the determinant of the resulting matrix is equal to K times the determinant of the original matrix.

iv) If each element of a row (column) of a square matrix is the sum of two terms then its determinant can be expressed as the sum of two determinants of two square matrices of the same order.

$$\begin{vmatrix} a_1+x & a_2 & a_3 \\ b_1+y & b_2 & b_3 \\ c_1+z & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & a_2 & a_3 \\ y & b_2 & b_3 \\ z & c_2 & c_3 \end{vmatrix}$$

Ex 1 : $$\begin{vmatrix} a_1+x & a_2+l & a_3 \\ b_1+y & b_2+m & b_3 \\ c_1+z & c_2+n & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & l & a_3 \\ b_1 & m & b_3 \\ c_1 & n & c_3 \end{vmatrix} + \begin{vmatrix} x & a_2 & a_3 \\ y & b_2 & b_3 \\ z & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & l & a_3 \\ y & m & b_3 \\ z & n & c_3 \end{vmatrix}$$

Ex : 2 : $$\begin{vmatrix} a_1+x & a_2+l & a_3+p \\ b_1+y & b_2+m & b_3+q \\ c_1+z & c_2+n & c_3+r \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & p \\ b_1 & b_2 & q \\ c_1 & c_2 & r \end{vmatrix} + \begin{vmatrix} a_1 & l & a_3 \\ b_1 & m & b_3 \\ c_1 & n & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} a_1 & l & p \\ b_1 & m & q \\ c_1 & n & r \end{vmatrix} + \begin{vmatrix} x & a_2 & a_3 \\ y & b_2 & b_3 \\ z & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & a_2 & p \\ y & b_2 & q \\ z & c_2 & r \end{vmatrix} + \begin{vmatrix} x & l & a_3 \\ y & m & b_3 \\ z & n & c_3 \end{vmatrix} + \begin{vmatrix} x & l & p \\ y & m & q \\ z & n & r \end{vmatrix}$$

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v) If the elements of a row (column) of a square matrix are added with K times the corresponding elements of any other row (column) then the value of the determinant of the resulting matrix is unaltered

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 + ky_1 & y_1 & z_1 \\ x_2 + ky_2 & y_2 & z_2 \\ x_3 + ky_3 & y_3 & z_3 \end{vmatrix}$$

vi) The sum of the products of the elements of any row (column) of a squre matrix with the cofactors of the corresponding elements of any other row (column) is zero.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\Rightarrow$ $a_1 A_2 + b_1 B_2 + c_1 C_2 = 0$
$a_1 A_3 + b_1 B_3 + c_1 C_3 = 0$
$a_2 A_1 + b_2 B_1 + c_2 C_1 = 0$

vii) If the elements of a square matrix are Poly nomials in x and two rows (columns) be come identical when $x = a$ then $x–a$ is a factor of its determinant and if three rows are identical then $(x–a)^2$ is a factor

Note:

i) $|A| = |A^T|, |AB| = |A||B| = |B||A|$

ii) $|KA| = K^n |A|$, n = order of A.

iii) If $\Delta = |a_{ij}|$ is a determinant of order n, then the value of the determinant $[A_{ij}]$, where A_{ij} is the cofactor of a_{ij} is, Δ^{n-1}.

iv) Determinant of nilpotent matrix is 0

v) Determinant of an orthogonal matrix = 1 or –1

vi) Determinant of a Skew - symmetric matrix of odd order is 0.

vii) Determinant of Hermitian matrix is purely real.

viii) Determinant of triangular matrix is zero

ix) The determinant of a unit matrix is 1

x) $|AB| = 0 \Rightarrow |A| = 0$ (or) $|B| = 0$

6. Standard results

i) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

ii) $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

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iii) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)\,(b-c)\,(c-a)$

iv) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} = (a+b+c)\,(a-b)\,(b-c)\,(c-a)$

v) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (ab + bc + ca)\;\;(a-b)\,(b-c)\,(c-a)$

vi) $\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)\left(a^2+b^2+c^2+ab+bc+ca\right)$

vii) $\begin{vmatrix} x+a_{11} & a_{12} & a_{13} \\ a_{21} & x+a_{22} & a_{23} \\ a_{31} & a_{32} & x+a_{33} \end{vmatrix} = x^3 + (a_{11}+a_{22}+a_{33})x^2$

$$+\left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\} x + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

viii) $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$

ix) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

x) $\begin{vmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} = 1+a^2+b^2+c^2$

xi) $\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix} = -8$

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xii) $\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+3)^2 & (n+4)^2 & (n+5)^2 \\ (n+6)^2 & (n+7)^2 & (n+8)^2 \end{vmatrix} = -216$

7. Product of Determinants of the same Order :

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ *and* $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then row by row multiplication of Δ_1 *and* Δ_2 is given by

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Note: Multiplication can also be performed row by column; column by row or column by column as required in the problem.

Ex :1 : $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \begin{vmatrix} \cos P & \cos Q & \cos R \\ \sin P & \sin Q & \sin R \\ 0 & 0 & 0 \end{vmatrix}$

Ex – 2 : $\begin{vmatrix} (a+x)^2 & (a+y)^2 & (a+z)^2 \\ (b+x)^2 & (b+y)^2 & (b+z)^2 \\ (c+x)^2 & (c+y)^2 & (c+z)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 1 & 2a \\ b^2 & 1 & 2b \\ c^2 & 1 & 2c \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$

8. Derivative of Determinants :

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where $f_1(x), f_2(x), g_1(x)$ *and* $g_2(x)$ are the functions of x. then,

$$\Delta^1(x) = \begin{vmatrix} f_1^1(x) & g_1^1(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f^1{}_2(x) & g^1{}_2(x) \end{vmatrix}$$

Also

$$\Delta^1(x) = \begin{vmatrix} f_1^1(x) & g_1(x) \\ f_2^1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1^1(x) \\ f_2(x) & g_2^1(x) \end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

9. Integration of Determinants :

If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$, then $\int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$

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Here, $f(x)$ *and* $g(x)$ are functions of x and λ_1, λ_2 are constants.

Note: This formula is only applicable, if there is a variable only in one row or column, otherwise expand the determinant and then integrate.

10. Singular and non-singular matrix :

If the determinant of a square matrix is zero then it is called a singular matrix otherwise non-singular matrix.

Note: i) A is singular $\Leftrightarrow A^T$ is singular A is non-singular $\Leftrightarrow A^T$ is non-singular

ii) If A and B are non-singular matrices of the same type, then AB is non-singular of the same type.

iii) If product of two non-zero square matrices is a zero matrix, then both of them must be singular matrices.

iv) Every orthogonal matrix is non -singular.

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