3 ALGEBRA OF MATRICES

SYNOPSIS

1. Matrix :

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by open ( ) or closed [ ] brackets. The numbers are called the elements of the matrix or entries of the matrix.

2. Order of Matrix : A matrix having 'm' rows and 'n' columns is called a matrix of order '$m \times n$' or simply '$m \times n$' matrix (read as m by n matrix). A matrix A of order '$m \times n$' is usually written in the following manner

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1j} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2j} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots & \ldots & a_{ij} & \ldots & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & a_{mj} & \ldots & \ldots & a_{mn} \end{bmatrix}_{m \times n}$$

In a compact form the above matrix is represented by

$A = \left[a_{ij}\right]_{m \times n}, 1 \le i \le m, 1 \le j \le n \,(or)\, A = \left[a_{ij}\right]$ The numbers $a_{11}, a_{12}, \ldots\ldots$, etc. are known as the elements of the matrix A. The element a_{ij} in the matrix A belongs to i^{th} row and j^{th} column.

Note: A matrix of order '$m \times n$' contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

Eg : Order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2×3

3. Types of matrices:

i) Row matrix: A matrix is said to be a row matrix if it has only one row and any number of columns.

Eg : $\begin{bmatrix} 5 & 0 & 3 \end{bmatrix}$ is a row matrix of order 1×3

ii) Column matrix : A matrix is said to be a column matirx if it has only one column and any number of rows.

Eg : $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3×1

iii) Singleton matrix : If in a matrix there is only one element then it is called singleton matrix.

Thus, $A = \left[a_{ij}\right]_{m \times n}$ is a singleton matrix, if m = n = 1

Eg : $[2], [3], [a], [-3]$ are singleton matrices.

iv) Null matrix (or) Zero matrix : If in a matrix all the elements are zeros then it is called a zero matrix and it is generally denoted by O. Thus $A = \left[a_{ij}\right]_{m \times n}$ is a zero matrix if $a_{ij} = 0 \,\forall$ i and j.

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Eg : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices, but of different orders.

v) Rectangular matrix : In a matrix if the number of rows is not equal to the number of columns then the matrix is called a rectangular matrix.

Eg : $\begin{bmatrix} 3 & -2 \\ 5 & 0 \\ 7 & 1 \end{bmatrix}_{3\times 2}$

vi) Square matrix : If number of rows and number of columns in a matrix are equal, then it is called a square matrix.

Eg : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3.

Note : An $n \times n$ square matrix is also called a square matrix of order n.

vii) Principal diagonal of a square matrix : In a square matrix the diagonal from first element of the first row to the last element of the last row is called the principal diagonal of the square matrix.

Eg : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then a_{11}, a_{22}, a_{33} constitutes diagonal of A

viii) Diagonal matrix : In a square matrix if all the elements outside the principal diagonal are zeros, the elements of principal diagonal may or may not be zero, then the matrix is said to be diagonal matrix.

Eg : $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ is a diagonal matrix of order 3×3, which can be denoted by diag $[d_1, d_2, d_3]$

ix) Identity matrix or Unit matrix :

A square matrix in which each element in the principal diagonal is '1' and rest are all zeros is called an identity matrix or unit matrix. Thus, the square matrix $A = [a_{ij}]$ is an identity matrix,

if $a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{if } i \neq j \end{cases}$

We denote the identity matrix of order n by I_n.

Eg : $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 2 and 3 respectively.

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x) Scalar matrix : A square matrix whose all non diagonal elements are zeros and diagonal elements are equal is called a scalar matrix. Thus, if $A=\left[a_{ij}\right]$ is a square matrix and $a_{ij}=\begin{cases}\alpha, \text{if } i=j\\ 0, \text{ if } i\neq j\end{cases}$, then A is a scalar matrix.

Eg : $\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}, \begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$

Unit matrix and null square matrices are also scalar matrices.

xi) Triangular matrix : A square matrix $\left[a_{ij}\right]$ is said to be triangular matrix if each element above or below the principal diagonal is zero.

xii) Upper triangular matrix: A square matrix $\left[a_{ij}\right]$ is called the upper triangular matrix, if $a_{ij}=0$ when $i>j$.

Eg : $\begin{bmatrix}3 & 1 & 2\\ 0 & 4 & 3\\ 0 & 0 & 6\end{bmatrix}$ is an upper triangular matrix of order 3 × 3.

xiii) Lower triangular matrix: A square matrix $\left[a_{ij}\right]$ is called the lower triangular matrix, if $a_{ij}=0$ when $i<j$.

Eg : $\begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 4 & 5 & 2\end{bmatrix}$ is a lower triangular matrix of order 3 × 3

4. Trace of a matrix : The sum of diagonal elements of a square matrix A is called the trace of matrix A, which is denoted by tr (A). Trace is also called as spur.

i.e., tr $(A)=\sum_{i=1}^{n}a_{ii}=a_{11}+a_{22}+.....a_{nn}$

5. Properties of trace of a matrix :

Let $A=\left[a_{ij}\right]_{n\times n}$ and $B=\left[b_{ij}\right]_{n\times n}$ and λ be a scalar

i) $tr\left(\lambda A\right)=\lambda tr\left(A\right)$

ii) $tr\left(A\pm B\right)=tr\left(A\right)\pm tr\left(B\right)$

iii) $tr\left(AB\right)=tr\left(BA\right)$

iv) $tr\left(I_n\right)=n$

v) $tr\left(AB\right)\neq tr\left(A\right).tr\left(B\right)$

vi) If A,B,C are square matrices of order n, then Tr (ABC) = Tr (BCA) = Tr (CAB)

6. Equality of matrices : Two matrices A and B are said to be equal, if

i) A and B are of the same type (order)

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ii) the corresponding elements of A and B are equal.

Eg : $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 0 & 1 \\ -1 & -3 & 5 \end{bmatrix}$ are equal iff a = 2, b = 0, c = 1, d = -1, e = -3, f = 5

Note : Inequality never exist between matrices A,B i.e., statements like $A > B$ or $A < B$ are meaningless

7. Addition of matrices :

If A and B are two matrices of the same type, then their sum denoted by A+B is defined to be the matrix of the same type and is obtained by adding the corresponding elements of A and B.

i.e., If $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$ then $A+B=[c_{ij}]_{m\times n}$ where $c_{ij}=a_{ij}+b_{ij}$

8. Properties on addition of matrices :

i) A+B = B+A (commutative law)

ii) (A+B)+C = A+(B+C) (Associative law)

iii) A+O=O+A = A where O is zero matrix which is called additive identity.

iv) $A+(-A)=O=(-A)+A$, where $(-A)$ is obtained by changing the sign of every element of A, which is additive inverse of the matrix A.

v) If $A + B = O$, then $B = -A$ or $A = -B$

vi) If $A + B = A + C$, then $B = C$

vii) If $A + B = C$, then $A = C - B$.

9. Difference of two matrices : If A and B are two matrices of the same type then A-B is defined as A+(-B)

Note: If two matrices A and B are of different orders, we can not define A+B and A-B.

10. Multiplication of a matrix by a scalar : Let $A=\left[a_{ij}\right]_{m\times n}$ be a matrix and k be a number, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.

Thus, if $A=\left[a_{ij}\right]_{m\times n}$, then $kA=Ak=\left[ka_{ij}\right]_{m\times n}$.

11. Properties of scalar multiplication :

If A, B are matrices of the same order and λ, μ are any two scalars then

i) $\lambda(A+B)=\lambda A+\lambda B$

ii) $(\lambda+\mu)A=\lambda A+\mu A$

iii) $\lambda(\mu A)=(\lambda\mu A)=\mu(\lambda A)$

iv) $(-\lambda A)=-(\lambda A)=\lambda(-A)$

v) $0A = O$ (where 0 is number, O is null matrix)

vi) $\alpha O = O$ (O is a null matrix).

vii) $(-1)A = -A$ and $-A$ is called the negative of the matrix A. Also $-(-A) = A$.

12. Multiplication of matrices : Two matrices A and B are conformable for the product AB if the number of columns in A is same as the number of rows in B thus, if

$A=\left[a_{ik}\right]_{m\times p}$, $B=\left[b_{kj}\right]_{p\times n}$ then

$AB=\left[c_{ij}\right]_{m\times n}$ where $c_{ij}=\sum_{k=1}^{p}a_{ik}b_{kj}$

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13. Properties of matrix multiplication :

If A,B and C are three matrices such that their product is defined, then

i) $AB \neq BA$,(Generally not commutative)

ii) $(AB)C = A(BC)$ =ABC, (Associative law)

iii) $IA = A = AI$, where I is identitiy matrix for matrix multiplication.

iv) $A(B+C) = AB + AC$, (Distributive law)

v) (A + B).C = AC + BC, (Distributive law)

vi) If $AB = O$, then either A or B need not be equal to O.

vii) If AB = AC then B need not be equal to C even if $A \neq O$.

viii) $A^2 = A\,.\,A$; $A^n = A.A.A.....n$ times

ix) $(kA)B = k(AB) = A(kB)$ and $(kA)(lB) = kl(AB)$ where k, l are scalars.

14. Remember that if A and B are two square matrices of the same order, then

i) $(A+B)^2 = A^2 + B^2 + AB + BA$

ii) $(A-B)^2 = A^2 + B^2 - AB - BA$

iii) $A^m A^n = A^{m+n}$

iv) $(A^m)^n = A^{mn} = (A^n)^m$

v) $I^n = I \quad \forall n \in N$

vi) If A = diag$(d_1, d_2, d_3 ... d_n)$ then $A^n = diag\left(d_1^{\,n}, d_2^{\,n}, d_3^{\,n} .....d_n^{\,n}\right)$

vii) If a square matrix, which is commutative with every square matrix of the same order for multiplication then it is necessarily a scalar matrix.

15. Commute : Two matrices A and B are Commute if $AB = BA$.

Let A, B are two square matrices which are commute then i.e. AB = BA then

i) $(A+B)^2 = A^2 + 2AB + B^2$

ii) $(A-B)^2 = A^2 - 2AB + B^2$

iii) $(A+B)\,(A-B) = A^2 - B^2$

iv) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

v) $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

vi) $(A+B)\,(A^2 - AB + B^2) = A^3 + B^3$

vii) $(A-B)\,(A^2 + AB + B^2) = A^3 - B^3$

16. Transpose of a matrix : The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of matrix A and is denoted by A^T *or* A'

From the definition it is obvious that if order of A is $m \times n$, then order of A^T is $n \times m$

17. Properties of transpose of a matrix :

Let A and B be two matrices then,

i) $(A^T)^T = A$

ii) $(A \pm B)^T = A^T \pm B^T$,A and B being of the same order

iii) $(kA)^T = kA^T$, k be any scalar

iv) $(AB)^T = B^T A^T$, A and B being conform able for the product AB

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v) $\left(A_1A_2A_3....A_{n-1}A_n\right)^T = A_n{}^T A_{n-1}^T \ ....A_3{}^T A_2{}^T A_1{}^T$

vi) A=B $\Leftrightarrow A^T = B^T$

18. Symmetric matrix: A square matrix $A=\left[a_{ij}\right]$ is called symmetric matrix if $a_{ij} = a_{ji} \ \forall \ i, j$ *i.e.*, $A^T = A$

Eg : $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

19. Skew-Symmetric matrix : A square matrix $A=\left[a_{ij}\right]$ is called skew -symmetric matrix if

$a_{ij} = -a_{ji} \ \forall$ i, j i.e., $A^T = -A$.

Eg : $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Note: i) All principal diagonal elements of a skew-symmetric matrix are always zeros

ii) Trace of a skew-symmetric matrix is zero

20. Properties of symmetric and Skew-symmetric matrices :

i) If A is a square matrix, then $A + A^T, AA^T, A^T A$ are symmetric matrices

ii) If A is square matrix then $A - A^T$ is a skew-symmetric matrix.

iii) If A is a symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^T AB$ are also symmetric matrices, where $n \in N$, $K \in R$ and B is a square matrix of order that of A

iv) If A is a skew-symmetric matrix, then

a) A^{2n} is symmetric matrix for $n \in N$

b) A^{2n+1} is a skew-symmetric matrix for $n \in N$

c) kA is also skew-symmetric matrix, where $k \in R$

v) If A, B are two symmetric matrices, then

a) $A \pm B, AB + BA$ are also symmetric matrices,

b) AB-BA is a skew-symmetric matrix,

c) AB is a symmetric matrix, when AB = BA

vi) If A, B two skew-symmetric matrices, then

a) $A \pm B$,AB-BA are skew-symmetric matrices

b) AB+BA is a symmetric matrix.

vii) a) If A is a skew-symmetric matrix and B is a square matrix of order that of A then $B^T AB$ is also skew-symmetric matrix.

b) If A is a skew-symmetric matrix and C is a column matrix, then C^T AC is a zero matrix.

viii) Every square matrix A can be uniquely expressed as sum of a symmetric and skew-symmetric matrices of same order

ix) If A,B are symmetric matrices of same order and $X = AB + BA$, $y = AB - BA$ then $XY + YX = 0$

i.e., $A = \left[\frac{1}{2}\left(A + A^T\right)\right] + \left[\frac{1}{2}\left(A - A^T\right)\right]$.

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Note: If a matrix A is both symmetric and skew-symmetric then A is null matrix.

21. Orthogonal matrix

Definition : A is square matrix such that $AA^T = A^TA = I$ then A is called orthogonal matrix.

Ex. $A = \frac{1}{2}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, $B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

clearly $AA^T = A^TA = I$

$BB^T = B^TB = I$

$\therefore$ A, B are orthogonal matrices

Geometrical interpretation of an orthogonal matrix

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Treating each row as a row vector i.e., $\begin{aligned} \overline{a} &= a_1 i + b_1 j + c_1 k \\ \overline{b} &= a_2 i + b_2 j + c_2 k \\ \overline{c} &= a_3 i + b_3 j + c_3 k \end{aligned}$

we have, $A = \begin{bmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \end{bmatrix}_{3\times 1}$

$$AA^T = I \Rightarrow \begin{bmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \end{bmatrix}_{3\times 1} [\overline{a} \ \ \overline{b} \ \ \overline{c}]_{1\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$\Rightarrow \begin{bmatrix} \overline{a}\cdot\overline{a} & \overline{a}\cdot\overline{b} & \overline{c}\cdot\overline{c} \\ \overline{b}\cdot\overline{a} & \overline{b}\cdot\overline{b} & \overline{b}\cdot\overline{c} \\ \overline{c}\cdot\overline{a} & \overline{c}\cdot\overline{b} & \overline{c}\cdot\overline{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \overline{a}\cdot\overline{a} = \overline{b}\cdot\overline{b} = \overline{c}\cdot\overline{c} = 1$$

and $\overline{a}\cdot\overline{b} = \overline{b}\cdot\overline{c} = \overline{c}\cdot\overline{a} = 0$

similarly, $A^TA = I$ will have a same meaning

Thus $AA^T = A^TA = I$

i.e., A is orthogonal if 3 row vectors (column vectors) constitute an orthonormal traid of unit vectors.

22. Properties of Orthogonal matrix :

i) If A is orthogonal, then A^T is also orthogonal.

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ii) If A and B are orthogonal matrices of same order then AB and BA are also orthogonal.

iii) The sum of the squares of elements of any row or column of an orthogonal matrix is 1.

iv) The sum of the products of the corresponding elements of any two rows or columns is 0

23. Special types of matrices :

i) Idempotent matrix: A square matrix A is called an idempotent matrix if $A^2 = A$

Eg : A= $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix.

Properties :

a) If A, B are two idempotent matrices and AB = BA = O then A+B is idempotent matrix.

b) If A is idempotent matrix then I-A is also idempotent

c) Every non-singular idempotent matrix is always unit matrix

d) If AB=A , BA=B then A and B are idempotent matrices and $A^n + B^n = A + B$

ii) Involutory matrix: A square matrix A is called an involutory matrix if $A^2 = I$

Eg : $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an involutory matrix.

iii) Nilpotent matrix: A square matrix A is called a nilpotent matrix if there exists atleast one $p \in N$ such that $A^p = O$, where the least value p is called index of the nilpotent matrix A

Eg : A= $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3.

Note: Trace of a nilpotent matrix is zero.

iv) Periodic matrix: A matrix A is called a periodic matrix if $A^{k+1} = A$ where k is a positive integer. The least value of k is said to be period of A

v) Conjugate of a matrix: The matrix obtained from any given matrix A containing complex numbers as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$.

Eg : $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$

vi) Transposed conjugate of a matrix: The transpose of the conjugate of a matrix A is called trans posed conjugate of A and is denoted by A^θ or A^*. The conjugate of the transpose of A is the same as the transpose of the conjugate of A

i.e., $\left(\overline{A^T}\right) = \left(\overline{A}\right)^T = A^\theta$.

Eg : $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then $A^\theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

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vii) Hermitian matrix: A square matrix 'A' is said to be Hermitian matrix if $A^{\theta} = A$

Eg : $\begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ is Hermitian matrix.

Note: All the principal diagonal elements of herimitian matrix are real.

viii) Skew Hermitian matrix : A square matrix A is said to be a skew-Hermitian if $A^{\theta} = -A$

Eg : $\begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$ is a skew-hermitian matrix.

Note: All the principal diagonal elements of skew-hermitian matrix are either zero or purely imaginary.

ix) Unitary matrix: A square matrix A is said to be unitary, if $A^{\theta}A = I = AA^{\theta}$

The determinant of unitary matrix is one and it is non-singular.

24. i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^2 - (a+d)A + (ad-bc)I = O$

ii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$A^3 - (a_{11}+a_{22}+a_{33})A^2 + \left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\} A - |A| I = O$$

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