

RUNNING NOTES

Functions

Teacher's handwritten notes, typed

IN THESE NOTES

Functions

Image – pre – image

Domain, Codomain, Range

Polynomial function

constant function

Identity function

Linear function

Quadratic function

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Functions

Def: A relation f from A to B is said to be a function if each and every element in A is associated to a unique element in B . It is denoted by $f : A \rightarrow B$ (Read it as f maps A to B)

Ex [diagram: arrow diagram f from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d\}$; $1 \rightarrow a, 2 \rightarrow a, 3 \rightarrow b, 4 \rightarrow c$]

$f : A \rightarrow B$ is a function.

Ex [diagram: arrow diagram g from $B = \{1, 2, 3\}$ to $C = \{a, b, c, d\}$; $1 \rightarrow a, 2 \rightarrow a, 3$ not mapped]

g from B to C is not a function

Ex [diagram: arrow diagram h from $C = \{1, 2, 3\}$ to $D = \{a, b, c, d\}$; $1 \rightarrow a, 1 \rightarrow b, 2 \rightarrow c, 3$ joined to d]

h from C to D is not a function

Image – pre – image

If $f : A \rightarrow B$ is a function and $(a, b) \in f$ then b is called image of a and a is called preimage of b

[diagram: arrow diagram f from A to B with a joined to b]

Domain, Codomain, Range

If $f : A \rightarrow B$ is a function then A is called Domain of f and B is called codomain of f

The set of all f -images of the elements of A in B is called Range of f

It is denoted by $f(A)$

Ex [diagram: arrow diagram f from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d\}$]

Domain = $A = \{1, 2, 3, 4\}$

codomain = $B = \{a, b, c, d\}$

Range of $f = f(A) = \{a, b, c\}$

Polynomial function

If $f : R \rightarrow R$ is defined by

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$ where $a_0 \neq 0$,

$a_1, a_2, \dots, a_n \in R, n \in W$ is a polynomial

function of degree n in x

Ex: $f(x) = x^2 + 3x + 5$ is a poly. function of degree 2.

Note If the domain of a function is not given then its Domain is taken as the set of all real values of x for which $f(x)$ is define.

Ex $f(x) = 3x + 5$, Domain = $\mathbb{R} = (-\infty, \infty)$

Note Domain of a polynomial function is $\mathbb{R}$.

constant function:

A function $f : A \rightarrow B$ is a constant function if the range of f contains only one element.

Ex: [diagram: arrow diagram f from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c\}$; all of $1, 2, 3, 4 \rightarrow a$]

Range of $f = \{a\}$

Graph of constant function is a line parallel to x-axis

Ex $y = f(x) = 2$

[graph: x-axis ($y = 0$) and y-axis; dashed horizontal lines $y = 2$ ✓ and $y = 1$; points $(1, 2)$ and $(2, 2)$ marked on $y = 2$]

Identity function:

let A be a nonempty set then

$f : A \rightarrow A$ defined by $f(x) = x, \forall x \in A$ is

called the identity function on A and it is denoted by I_A .

Ex [diagram: arrow diagram I_A from $A = \{1, 2, 3, 4\}$ to $A = \{1, 2, 3, 4\}$; each element mapped to itself]

Graph of identity function is a straight line passing through origin and inclined at an angle of 45° with x-axis

[graph: line $y = x$ through origin at 45° to x-axis, passing through $(-1, -1), (1, 1), (2, 2)$]

$$y = f(x) = x$$

$$x = 1, y = 1$$

$$x = 2, y = 2$$

$$x = 0, y = 0$$

$$x = -1, y = -1$$

Linear function:

let $y = f(x) = ax + b$ ($a \neq 0$) is a

linear function. It represent a straight line

Ex $y = x + 2$ is a linear function

put $x = 0 \Rightarrow y = 2 \rightarrow$ y-intercept $(0, 2)$

$y = 0 \Rightarrow x = -2 \rightarrow$ x-intercept $(-2, 0)$

[graph: straight line through $(-2, 0)$ and $(0, 2)$]

Note $y = f(x) = ax + b$

(i) $a > 0$, Graph of $f(x)$ is a straight line whose inclination is less than 90°

[graph: rising straight line crossing the x-axis with an acute angle marked]

(ii) $a < 0$, Graph of $f(x)$ is a straight line whose inclination is greater than 90°

[graph: falling straight line crossing the x-axis with an obtuse angle marked]

Quadratic function:

let $y = f(x) = ax^2 + bx + c$ ($a \neq 0$) is called
a Quadratic function (or) Quadratic expression

■ Quadratic equation:

The equation $ax^2 + bx + c = 0$ ($a \neq 0$) is
called a Quadratic equation.

Note: The roots of Q.E $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \text{ where } \Delta = b^2 - 4ac \text{ (} \rightarrow \text{ Discriminant)}$$

$\Delta > 0 \Rightarrow$ Roots are real & distinct

$\Delta = 0 \Rightarrow$ Roots are real & equal

$\Delta < 0 \Rightarrow$ No real roots (Imaginary)

Note If α, β are the roots of $ax^2 + bx + c = 0$

then (i) sum of roots $= \alpha + \beta = -\frac{b}{a} = -\frac{x \text{ co - eff.}}{x^2 \text{ co - eff.}}$

(ii) product of roots $= \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{x^2 \text{ co - effi.}}$

V.Imp (iii) Difference of roots $= |\alpha - \beta| = \frac{\sqrt{\Delta}}{|a|} = \sqrt{\frac{b^2 - 4ac}{|a|}}$

Hint $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$$

$$|\alpha - \beta| = \sqrt{\frac{b^2 - 4ac}{|a|}}$$

Result If α, β are the roots of $ax^2 + bx + c = 0$

then $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

proof $ax^2 + bx + c = a\left[x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a}\right]$

$$= a\left[x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\right]$$

$$= a[x^2 - (\alpha + \beta)x + \alpha\beta]$$

$$= a[x^2 - \alpha x - \beta x + \alpha\beta]$$

$$= a(x - \alpha)(x - \beta)$$

Graph of $y = ax^2 + bx + c$

let $y = ax^2 + bx + c \rightarrow$ General form

$$= a\left[x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \rightarrow \text{vertex form}$$

(i) If $a > 0$ then $y = ax^2 + bx + c$ represents

an upward parabola with vertex $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

It attains minimum value at $x = -\frac{b}{2a}$

and that minimum value is $\frac{4ac - b^2}{4a}$

$$\therefore \forall x \in \mathbb{R}, \text{Range} = \left[\frac{4ac - b^2}{4a}, \infty\right)$$

[graph: upward parabola cutting x-axis at two points, vertex at $x = -\frac{b}{2a}, y = \frac{4ac - b^2}{4a}$; labelled $a > 0, \Delta > 0$]

[graph: upward parabola touching x-axis; labelled $a > 0, \Delta = 0$]

[graph: upward parabola above x-axis; labelled $a > 0, \Delta < 0$]

(ii) If $a < 0$, then $y = ax^2 + bx + c$ represents

a downward parabola with vertex $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

It attains maximum value at $x = -\frac{b}{2a}$ and

that maximum value is $\frac{4ac - b^2}{4a}$

$$\therefore \forall x \in \mathbb{R}, \text{Range} = \left(-\infty, \frac{4ac - b^2}{4a}\right]$$

[graph: downward parabola cutting x-axis at two points, vertex at $x = -\frac{b}{2a}, y = \frac{4ac - b^2}{4a}$; labelled $a < 0, \Delta > 0$]

[graph: downward parabola touching x-axis at $-\frac{b}{2a}$; labelled $a < 0, \Delta = 0$]

[graph: downward parabola below x-axis, vertex at $-\frac{b}{2a}, \frac{4ac - b^2}{4a}$; labelled $a < 0, \Delta < 0$]

observation:

$$y = f(x) = a(x - \alpha)(x - \beta) \rightarrow \text{Factored form}$$

(i) $a > 0, \Delta > 0$

[graph: upward parabola cutting x-axis at α and β , minimum at $(\frac{\alpha + \beta}{2}, f(\frac{\alpha + \beta}{2}))$]

(ii) $a < 0, \Delta > 0$

[graph: downward parabola cutting x-axis at α and β , maximum $f(\frac{\alpha + \beta}{2})$ at $x = \frac{\alpha + \beta}{2}$]

If $y = a(x - \alpha)^2 \rightarrow$ perfect square

(i) $a > 0$

[graph: upward parabola touching x-axis at $(\alpha, 0)$; $\Delta = 0$]

(ii) $a < 0$

[graph: downward parabola touching x-axis at $(\alpha, 0)$; $\Delta = 0$]

observation Vertex form

$$f(x) = y = ax^2 + bx + c \rightarrow \text{General form}$$

$$= a[x + \frac{b}{2a}]^2 + \frac{4ac - b^2}{4a} \rightarrow \text{vertex form}$$

$$y = a(x - h)^2 + k$$

$$\text{Vertex} = (h, k) = (-\frac{b}{2a}, \frac{4ac - b^2}{4a})$$

$$(\frac{4ac - b^2}{4a} \rightarrow f(-\frac{b}{2a}))$$

Graph of $y = (x - 2)(x - 3)$

[graph: upward parabola through $(0, 6)$, cutting x-axis at 2 and 3, vertex $(\frac{5}{2}, -\frac{1}{4})$, passing through $(5, 6)$]

$$\begin{aligned} f(\frac{5}{2}) &= (\frac{5}{2} - 2)(\frac{5}{2} - 3) \\ &= (\frac{1}{2})(-\frac{1}{2}) \\ &= -\frac{1}{4} \end{aligned}$$

put $x = 0$

$$\Rightarrow y = 6$$

$(0, 6) \rightarrow$ This is pt the curve cuts y-axis

Graph of $y = x^2 + 2x + 1 = (x + 1)^2$

[graph: upward parabola touching x-axis at $(-1, 0)$, passing through $(-2, 1)$ and $(0, 1)$]

$$x + 1 = 0 \Rightarrow x = -1$$

$$\Rightarrow y = 0$$

$(-1, 0)$

put

$$x = 0 \Rightarrow y = 1$$

Graph of $y = 2(x - 1)^2 + 3$

$\rightarrow$ upward parabola, vertex = $(1, 3)$

[graph: upward parabola with lowest point labelled $(1, 1)$, passing through $(0, 5)$ and $(2, 5)$]

Graph of $y = 4x^2 + 4x + 1 = (2x)^2 + 4x + 1^2 = (2x + 1)^2$

[graph: upward parabola touching x-axis at $(-\frac{1}{2}, 0)$, passing through $(-1, 1)$ and $(0, 1)$]

$$2x + 1 = 0$$

$$x = -\frac{1}{2}$$

$$\text{Vertex} = (-\frac{1}{2}, 0)$$

Graph of $y = x^2 + x + 1$

$$a > 0, \Delta = b^2 - 4ac = 1 - 4 = -3 < 0$$

$$\begin{aligned} \text{vertex} &= (-\frac{b}{2a}, \frac{4ac - b^2}{4a}) \text{ ((or) } f(-\frac{b}{2a})) \\ &= (-\frac{1}{2}, \frac{3}{4}) \end{aligned}$$

[graph: upward parabola above x-axis, vertex at $x = -\frac{1}{2}, y = \frac{3}{4}$, passing through $(-1, 1)$ and $(0, 1)$]

$$a = 1$$

$$b = 1$$

$$c = 1$$

[sketch: small pair of axes]

Graph of $y = x^2 - 5x + 6$

$$a > 0, \Delta = b^2 - 4ac = 25 - 24 = 1 > 0$$

$$x^2 - 5x + 6 = 0$$

$$\Rightarrow (x - 2)(x - 3) = 0 \text{ (or) } x = \frac{-b \pm \sqrt{D}}{2a}$$

$$\Rightarrow x = 2, 3 \text{ (2} \rightarrow \alpha, 3 \rightarrow \beta)$$

$$\frac{\alpha + \beta}{2} = \frac{5}{2}$$

$$f\left(\frac{\alpha + \beta}{2}\right) = \frac{25}{4} - \frac{25}{2} + 6 = \frac{25 - 50 + 24}{4} = -\frac{1}{4}$$

$$\text{Vertex} = \left(\frac{5}{2}, -\frac{1}{4}\right)$$

[graph: upward parabola through (0, 6), cutting x-axis at 2 and 3, vertex $(\frac{5}{2}, -\frac{1}{4})$, passing through (5, 6)]

$$a = 1$$

$$b = -5$$

$$c = 6$$

H.W Draw the graph

$$\textcircled{1} y = (x + 2)(x - 4)$$

$$\textcircled{2} y = (2x + 3)^2$$

$$\textcircled{3} y = 2(x - 2)^2 + 5$$

$$\textcircled{4} y = x^2 + 6x + 9$$

$$\textcircled{5} y = x^2 + 2x + 3$$

$$\textcircled{6} y = 2x^2 - 5x - 3$$

Solving Quadratic inequation

Ex $\textcircled{1} y = (x + 1)(x - 3) < 0 \Rightarrow x \in (-1, 3)$

$$\leq 0 \Rightarrow x \in [-1, 3]$$

$$> 0 \Rightarrow x \in (-\infty, -1) \cup (3, \infty)$$

$$\geq 0 \Rightarrow x \in (-\infty, -1] \cup [3, \infty)$$

[graph: upward parabola cutting x-axis at -1 and 3 (each marked = 0); sign + left of -1, - between, + right of 3; axis from $-\infty$ to ∞]

Ex $\textcircled{2} y = (2x - 1)^2 < 0 \Rightarrow x \in \phi$

$$\leq 0 \Rightarrow x \in \left\{\frac{1}{2}\right\}$$

$$> 0 \Rightarrow x \in \mathbb{R} - \left\{\frac{1}{2}\right\} = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

$$\geq 0 \Rightarrow x \in \mathbb{R} = (-\infty, \infty)$$

Sol [graph: upward parabola touching x-axis at $\frac{1}{2}$ (= 0); sign + on both sides; axis from $-\infty$ to ∞]

Ex 3: $y = (x - 1)^2 - 2 < 0 \Rightarrow x \in (1 - \sqrt{2}, 1 + \sqrt{2})$

$\leq 0 \Rightarrow x \in [1 - \sqrt{2}, 1 + \sqrt{2}]$

$> 0 \Rightarrow x \in (-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$

$\geq 0 \Rightarrow x \in (-\infty, 1 - \sqrt{2}] \cup [1 + \sqrt{2}, \infty)$

→ upward parabola,

vertex = $(1, -2)$

$y = 0$

$\Rightarrow (x - 1)^2 - 2 = 0$

$\Rightarrow (x - 1)^2 = 2$

$\Rightarrow x - 1 = \pm\sqrt{2}$

$\Rightarrow x = 1 \pm \sqrt{2}$ ✓

[graph: upward parabola cutting x-axis at two points, both labelled $1 + \sqrt{2}$ ($= 0$); sign + outside, - between; axis from $-\infty$ to ∞]

Ex $y = 9x^2 + 6x + 1 < 0 \Rightarrow x \in \phi$

$\leq 0 \Rightarrow x \in \{-\frac{1}{3}\}$

$> 0 \Rightarrow x \in (-\infty, -\frac{1}{3}) \cup (-\frac{1}{3}, \infty) = \mathbb{R} - \{-\frac{1}{3}\}$

$\geq 0 \Rightarrow x \in \mathbb{R}$

Sol $y = (3x + 1)^2$

Here $\Delta = 0$

[graph: upward parabola touching x-axis at $-\frac{1}{3}$ ($= 0$); sign + on both sides; axis from $-\infty$ to ∞]

$3x + 1 = 0$

$x = -\frac{1}{3}$

Ex $y = x^2 + 2x + 5 < 0 \Rightarrow x \in \phi$

$\leq 0 \Rightarrow x \in \phi$

$> 0 \Rightarrow x \in \mathbb{R}$

$\geq 0 \Rightarrow x \in \mathbb{R}$

Sol $a > 0, \Delta = b^2 - 4ac < 0$

[graph: upward parabola entirely above the x-axis; axis from $-\infty$ to ∞]

$\forall x \in \mathbb{R}, y > 0$

$a = 1$

$b = 2$

$c = 5$

-

$$\rightarrow y = 2x^2 + 3x - 5 < 0 \Rightarrow x \in \left(-\frac{5}{2}, 1\right)$$

$$\leq 0 \Rightarrow x \in \left[-\frac{5}{2}, 1\right]$$

$$> 0 \Rightarrow x \in \left(-\infty, -\frac{5}{2}\right) \cup (1, \infty)$$

$$\geq 0 \Rightarrow x \in \left(-\infty, -\frac{5}{2}\right] \cup [1, \infty)$$

Sol $a > 0, \Delta > 0$

$$2x^2 + 3x - 5 = 0$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-3 \pm \sqrt{49}}{4}$$

$$= \frac{-3 \pm 7}{4} = -\frac{5}{2}, 1$$

[graph: upward parabola cutting x-axis at $-\frac{5}{2}$ and 1 (each = 0); - between, + outside]

$$a = 2$$

$$b = 3$$

$$c = -5$$

$$\Delta = b^2 - 4ac$$

$$= 9 - 4(2)(-5)$$

$$= 9 + 40$$

$$= 49$$

Ex $y = (3 - x)(x - 2) < 0 \Rightarrow x \in (-\infty, 2) \cup (3, \infty)$

$$\leq 0 \Rightarrow x \in (-\infty, 2] \cup [3, \infty)$$

$$> 0 \Rightarrow x \in (2, 3)$$

$$\geq 0 \Rightarrow x \in [2, 3]$$

$$a < 0 \rightarrow \text{downward}$$

[graph: downward parabola cutting x-axis at 2 and 3 (each = 0); - outside, + between; axis from $-\infty$ to ∞]

Ex $y = -x^2 + 2x - 1 < 0 \Rightarrow x \in \mathbb{R} - \{1\}$

$$\leq 0 \Rightarrow x \in \mathbb{R}$$

$$> 0 \Rightarrow x \in \phi$$

$$\geq 0 \Rightarrow x \in \{1\}$$

$$y = -(x^2 - 2x + 1)$$

$$= -(x - 1)^2$$

$$a < 0 \rightarrow \text{downward parabola}$$

[graph: downward parabola touching x-axis at 1 (= 0); sign - on both sides; axis from $-\infty$ to ∞]

Ex $y = -2x^2 + x - 7 < 0 \Rightarrow x \in \mathbb{R}$

$\leq 0 \Rightarrow x \in \mathbb{R}$

$> 0 \Rightarrow x \in \phi$

$\geq 0 \Rightarrow x \in \phi$

$a < 0, \Delta = (1)^2 - 4(-2)(-7) < 0$

[graph: downward parabola entirely below the x-axis]

$\forall x \in \mathbb{R}, y < 0$

$y = -x^2 + 5x - 6 < 0 \Rightarrow x \in (-\infty, 2) \cup (3, \infty)$

$\leq 0 \Rightarrow x \in (-\infty, 2] \cup [3, \infty)$

$> 0 \Rightarrow x \in (2, 3)$

$\geq 0 \Rightarrow x \in [2, 3]$

$a < 0, \Delta = 25 - 24 = 1 > 0$

$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm 1}{-2} = 2, 3$

[graph: downward parabola cutting x-axis at 2 and 3 (each = 0); - outside, + between; axis from $-\infty$ to ∞]

If $\alpha < \beta$

Note and $(x - \alpha)(x - \beta) < 0 \Rightarrow x \in (\alpha, \beta)$ (less $\rightarrow$ lies)

$\leq 0 \Rightarrow x \in [\alpha, \beta]$

$> 0 \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$

$\geq 0 \Rightarrow x \in (-\infty, \alpha] \cup [\beta, \infty)$

[graph: upward parabola cutting x-axis at α and β ; + outside, - between; axis from $-\infty$ to ∞]

Ex $(x + 1)(x - 3) < 0 \Rightarrow x \in (-1, 3)$

$\leq 0 \Rightarrow x \in [-1, 3]$

$> 0 \Rightarrow x \in (-\infty, -1) \cup (3, \infty)$

$\geq 0 \Rightarrow x \in (-\infty, -1] \cup [3, \infty)$

Ex $(3 - x)(x - 5) < 0$ then $x \in \underline{\hspace{2cm}}$

| $\rightarrow (x - 3)(x - 5) > 0 \Rightarrow x \in (-\infty, 3) \cup (5, \infty)$

logarithms

$2^0 = 1$ then 0 is called logarithm of 1 to

the base 2

It can be write $0 = \log_2 1$

$2^1 = 2 \Rightarrow 1 = \log_2 2$

$$2^2 = 4 \Rightarrow 2 = \log_2 4$$

$$2^{-1} = \frac{1}{2} \Rightarrow -1 = \log_2 \left(\frac{1}{2}\right)$$

$$2^{-2} = \frac{1}{4} \Rightarrow -2 = \log_2 \left(\frac{1}{4}\right)$$

Note $a^x = N \Leftrightarrow x = \log_a N$

where $a > 0$, $a \neq 1$, $N > 0$, $x \in \mathbb{R}$

[diagram: number line from $-\infty$ to ∞ marking 0 and 1, with open circles at 0 and 1]

i.e $a \in (0, 1) \cup (1, \infty) = (0, \infty) - \{1\}$

$$N \in (0, \infty)$$

$$x \in (-\infty, \infty)$$

Remarks

$$0^2 = 0 \Rightarrow 2 = \log_0 0$$

$$0^3 = 0 \Rightarrow 3 = \log_0 0 \Rightarrow 2 = 3 \quad \text{X}$$

$$\therefore a \neq 0$$

$$1^2 = 1 \Rightarrow 2 = \log_1 1$$

$$1^3 = 1 \Rightarrow 3 = \log_1 1 \Rightarrow 2 = 3 \quad \text{X}$$

$$\therefore a \neq 1$$

$$(-1)^2 = 1 \Rightarrow 2 = \log_{-1} 1$$

$$(-1)^4 = 1 \Rightarrow 4 = \log_{-1} 1 \Rightarrow 2 = 4 \quad \text{X}$$

$$a < 0$$

$$\therefore a \in (0, 1) \cup (1, \infty)$$

Ex $2^4 = 16 \Leftrightarrow 4 = \log_2 16$

$$\log_2 x = 5 \Leftrightarrow x = 2^5$$

i.e $x = 32$

$$\log_5 (x - 1) = -1 \Leftrightarrow x - 1 = 5^{-1}$$

$$\begin{aligned} \text{i.e } x - 1 &= \frac{1}{5} \\ \Rightarrow x &= \frac{6}{5} \end{aligned}$$

Ex $\log_2 32 = ?$

Ans 5

$$\because 2^5 = 32$$

Ex $\log_{\frac{1}{5}} 25 = -2$

$$\left(\frac{1}{5}\right)^{-2} = (5^{-1})^{-2} = 5^2 = 25$$

Ex $\log_{\frac{1}{3}} 3 = -1$

Ex $\log_3 9 = 2$

Formulae:

① $\log_a 1 = 0 \because a^0 = 1$ where $a > 0, a \neq 1$

Ex $\log_3 1 = 0$

Ex $\log_1 1 = \text{undefined}$

Ex $\log_{-1} 1 = \text{undefined}$

② $\log_a a = 1 \because a^1 = a$ where $a > 0, a \neq 1$

Ex $\log_2 2 = 1$

$\log_{\frac{1}{5}} \left(\frac{1}{5}\right) = 1, \log_1 1 = \text{undefined}.$

③ $\log_a(mn) = \log_a m + \log_a n$

proof $\log_a m = x, \log_a n = y$

$$\Rightarrow m = a^x, n = a^y$$

$$\Rightarrow mn = a^x \cdot a^y$$

$$\Rightarrow mn = a^{x+y}$$

$$\Rightarrow \log_a(mn) = x + y$$

$$\Rightarrow \log_a(mn) = \log_a m + \log_a n$$

④ $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

Note $\log_a \left(\frac{mn}{pq}\right) = \log_a m + \log_a n - \log_a p - \log_a q$

⑤ $\log_{b^n} a^m = \left(\frac{m}{n}\right) \log_b a$

Note 1) $\log_b a^m = m \log_b a$

2) $\log_{b^n} a = \left(\frac{1}{n}\right) \log_b a.$

⑥ $\log_b a = \frac{1}{\log_a b}$

⑦ $\log_b a \cdot \log_a b = 1$

⑧ $\log_b a \log_c b = \log_c a$

note $\log_b a \log_c b \log_d c = \log_d a$

✓⑨ $\log_b a = \frac{\log a}{\log b} = \frac{\log_x a}{\log_x b}$

$\frac{\log a}{\log b} \neq \log\left(\frac{a}{b}\right)$

10) $a^{\log_a N} = N$

Ex $2^{\log_2 3} = 3$

11) $a^{\log_c b} = b^{\log_c a}$

Ex $2^{\log_3 5} = 5^{\log_3 2}$

12) If $\log_a m = \log_a n \Rightarrow m = n$

Graph of $\log_2 x$

let $y = f(x) = \log_2 x$

$x = 1, y = \log_2 1 = 0 \quad (1, 0)$

$x = 2, y = \log_2 2 = 1 \quad (2, 1)$

$x = 4, y = \log_2 4 = \log_2 2^2 = 2 \log_2 2 = 2(1) = 2 \quad (4, 2)$

$x = \frac{1}{2}, y = \log_2 \left(\frac{1}{2}\right) = \log_2 2^{-1} = (-1) \log_2 2 = (-1)(1) = -1 \quad \left(\frac{1}{2}, -1\right)$

$x = \frac{1}{4}, y = -2 \quad \left(\frac{1}{4}, -2\right)$

[graph: increasing log curve through (1, 0), (2, 1), (4, 2), dropping steeply near the y-axis below; axes marked -1 to 4 on x and -3 to 3 on y, with $\frac{1}{2}$ marked on x-axis]

Graph of $\log_{\frac{1}{2}} x$

let $y = f(x) = \log_{\frac{1}{2}} x$

$x = 1, y = \log_{\frac{1}{2}} 1 = 0 \quad (1, 0)$

$x = 2, y = \log_{\frac{1}{2}} 2 = \log_{2^{-1}} 2^1 = (1/-1) \log_2 2 = -1 \quad (2, -1)$

$x = 4, y = -2 \quad (4, -2)$

$x = \frac{1}{2}, y = \log_{\frac{1}{2}} \left(\frac{1}{2}\right) = 1 \quad \left(\frac{1}{2}, 1\right)$

$x = \frac{1}{4}, y = 2 \quad \left(\frac{1}{4}, 2\right)$

[graph: decreasing log curve, high near the y-axis, crossing x-axis at 1 and falling below as x increases; x-axis marked $\frac{1}{2}, 1, 2, 3, 4$; y-axis marked 1, 2, 3 and -1, -2, -3, -4]

Graph of $\log_a x$ ($a > 1$)

[graph: increasing curve through (1, 0), dropping steeply near the y-axis]

Graph of $\log_a x$ ($0 < a < 1$)

[graph: decreasing curve through (1, 0), rising steeply near the y-axis]

Inequalities

① $\log_a x > 0 \Leftrightarrow a > 1, x > 1$ (or) $0 < a < 1, 0 < x < 1$

[graph: $a > 1$ — increasing log curve through 1, region $x > 1$ marked]

[graph: $0 < a < 1$ — decreasing log curve through 1, region $0 < x < 1$ marked]

② $\log_a x \geq 0 \Leftrightarrow a > 1, x \geq 1$ (or) $0 < a < 1, 0 < x \leq 1$

③ $\log_a x < 0 \Leftrightarrow a > 1, 0 < x < 1$ (or) $0 < a < 1, x > 1$

④ $\log_a x \leq 0 \Leftrightarrow a > 1, 0 < x \leq 1$ (or) $0 < a < 1, x \geq 1$

Ex 1) $\log_2 3 > 0$

2) $\log_{\frac{1}{2}} \left(\frac{1}{3}\right) > 0$

3) $\log_{\frac{1}{2}} 3 < 0$

4) $\log_2 \left(\frac{1}{3}\right) < 0$

5) $\log_{0.3} 0.5 > 0$

6) $\log_2 0.5 < 0$

7) $\log_{3.2} 5.8 > 0$

8) $\log_{\frac{1}{5}} 100 < 0$

Ex $\log_{10}(x - 2) > 0$ then $x \in \underline{\hspace{2cm}}$

Sol $\log_{10}(x - 2) > 0$

$$\Rightarrow x - 2 > 1$$

$$\Rightarrow x > 3$$

$$\Rightarrow x \in (3, \infty)$$

① $\log_{512} 1 = 0$

② $\log_{512} 512 = 1$

③ If $\log_2 x + \log_2(x + 1) = 1$ then $x = \underline{\hspace{2cm}}$

→ $\log_2 x(x + 1) = 1$

$$\Rightarrow x(x + 1) = 2^1$$

$$\Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow x^2 + 2x - x - 2 = 0$$

$$\Rightarrow x(x + 2) - 1(x + 2) = 0$$

$$\Rightarrow x = 1 \text{ (or) } x = -2$$

For $x = -2$

$\log_2 x = \log_2 -2$ is undefined

$$\therefore x \neq -2$$

$$\therefore x = 1$$

④ Solve $\log_4(x^2 + x) - \log_4(x + 1) = 2$

Sol $\rightarrow \log_4\left(\frac{x^2 + x}{x + 1}\right) = 2$
 $\Rightarrow \log_4\left(\frac{x(x + 1)}{x + 1}\right) = 2$
 $\Rightarrow \log_4 x = 2$
 $\Rightarrow x = 4^2$
 $\Rightarrow x = 16$

⑤ $\log_3 x + \log_9 x^2 + \log_{27} x^3 = 3$

then $x = \underline{\hspace{2cm}}$

Sol $\log_3 x + \log_{3^2} x^2 + \log_{3^3} x^3 = 3$
 $\Rightarrow \log_3 x + \left(\frac{2}{3}\right) \log_3 x + \left(\frac{3}{3}\right) \log_3 x = 3$
 $\Rightarrow 3 \cdot \log_3 x = 3$
 $\Rightarrow \log_3 x = 1$
 $\Rightarrow x = 3^1$
 $\Rightarrow x = 3$

$$\log_b a^m = \left(\frac{m}{n}\right) \log_b a$$

⑥ $\frac{1}{1 + \log_a bc} + \frac{1}{1 + \log_b ac} + \frac{1}{1 + \log_c ab} = \underline{\hspace{2cm}}$

Sol $= \frac{1}{\log_a a + \log_a bc} + \frac{1}{\log_b b + \log_b ac} + \frac{1}{\log_c c + \log_c ab}$
 $= \frac{1}{\log_a abc} + \frac{1}{\log_b (abc)} + \frac{1}{\log_c (abc)}$
 $= \log_{abc} a + \log_{abc} b + \log_{abc} c$
 $= \log_{abc} abc$
 $= 1$

$$\log_b a = \frac{1}{\log_a b} \quad \checkmark$$

⑦ $\log_{y^2} x^3 \cdot \log_{z^2} y^3 \cdot \log_{x^2} z^3 = \underline{\hspace{2cm}}$

$$= \left(\frac{3}{2}\right)(\log_y x) \cdot \left(\frac{3}{2}\right)(\log_z y) \cdot \left(\frac{3}{2}\right)(\log_x z)$$

$$= \left(\frac{27}{8}\right) \log_y x \log_z y \log_x z$$

$$= \left(\frac{27}{8}\right)(1)$$

$$= \frac{27}{8}$$

$$\log_b a \log_a b = 1$$

$$\textcircled{8} \log_5 2 \cdot \log_{11} 5 \cdot \log_{32} 11 = \underline{\hspace{2cm}}$$

$$= \log_{32} 2$$

$$= \log_{2^5} 2^1$$

$$= \left(\frac{1}{5}\right) \log_2 2$$

$$= \left(\frac{1}{5}\right)(1) = \frac{1}{5}$$

$$\textcircled{10} (5.8)^{(\log_{5.8} 10) + 1} = \underline{\hspace{2cm}}$$

$$= 5.8^{\log_{5.8} 10} \cdot (5.8)^1$$

$$= 10(5.8)$$

$$= 58$$

$$a^{\log_a N} = N$$

$$a^{m+n} = a^m \cdot a^n$$

$$\textcircled{11} 2^{\log_4 25} = \underline{\hspace{2cm}}$$

$$= 2^{\log_2 5^2}$$

$$= 2^{\left(\frac{2}{2}\right) \log_2 5}$$

$$= 2^{\log_2 5} = 5$$

$$a^{\log_c b} = b^{\log_c a}$$

$$(\text{or}) 2^{\log_4 25}$$

$$= 25^{\log_4 2}$$

$$= 25^{\left(\frac{1}{2}\right) \log_2 2}$$

$$= 25^{\frac{1}{2}} = \sqrt{25} = 5$$

$$\textcircled{9} \frac{\log(7\sqrt{7}) + \log(3\sqrt{3}) - \log(5\sqrt{5})}{\log(4.2)} = \underline{\hspace{2cm}}$$

$$= \frac{\log\left(\frac{(7\sqrt{7})(3\sqrt{3})}{5\sqrt{5}}\right)}{\log 4.2}$$

$$= \frac{\log\left(\frac{7^{\frac{3}{2}} \cdot 3^{\frac{3}{2}}}{5^{\frac{3}{2}}}\right)}{\log 4.2} = \log_{4.2} \left(\frac{21}{5}\right)^{\frac{3}{2}}$$

$$= \log_{4.2} (4.2)^{\frac{3}{2}}$$

$$= \frac{3}{2}$$

$$\log_b a = \frac{\log a}{\log b}$$

$$7\sqrt{7} =$$

$$a\sqrt{a} = a^1 \cdot a^{\frac{1}{2}}$$

$$= a^{\frac{3}{2}}$$

$$\frac{\log a}{\log b} = \log_b a$$

⑫ Solve $\log_2(3x - 2) = \log_{\frac{1}{2}} x$

Sol $\log_2(3x - 2) = \log_{2^{-1}} x$

$$\log_2(3x - 2) = (-1) \log_2 x$$

$$\log_2(3x - 2) = \log_2 x^{-1}$$

$$\Rightarrow 3x - 2 = \frac{1}{x}$$

$$\Rightarrow 3x^2 - 2x - 1 = 0$$

$$\Rightarrow 3x^2 - 3x + x - 1 = 0$$

$$\Rightarrow 3x(x - 1) + 1(x - 1) = 0$$

$$\Rightarrow x = -\frac{1}{3} \text{ (or) } 1$$

for $x = -\frac{1}{3}$

$\log_{\frac{1}{2}} x$ is not defined

$$\therefore x = 1$$

Absolute value function (or) Modulus function

$$|-1| = -(-1) = 1$$

(Read it as mod of -1)

$$|0| = 0$$

$$|1| = 1$$

Note $|x| = \{ x, x \geq 0$

$$-x, x < 0 \}$$

Ex $|-2| = -(-2) = 2$ ($-2 \rightarrow -ve$)

$$|2| = 2$$
 ($2 \rightarrow +ve$)

Ex $|-3| = -(-3) = 3$ ($-3 \rightarrow -ve$)

$$|3| = 3$$
 ($3 \rightarrow +ve$)

Def: modulus function

$f(x) = |x|$ where $x \in \mathbb{R}$ is called

modulus function

Domain = $\mathbb{R}$, Range = $[0, \infty)$

Graph of $f(x) = |x|$

[graph: V-shaped graph with vertex at origin, arms through $(\pm 1, 1)$, $(\pm 2, 2)$, $(\pm 3, 3)$; axes marked -3 to 4 on x and 1 to 3 on y]

$$y = f(x) = |x|$$

$$x = 0, y = |0| = 0$$

$$x = 1, y = |1| = 1$$

$$x = 2, y = |2| = 2$$

$$x = -1, y = 1$$

properties

$$\textcircled{1} |x| \geq 0, \forall x \in \mathbb{R}$$

$$\textcircled{2} |x| = 0 \Rightarrow x = 0$$

$$\text{** } \textcircled{3} |x| = x \Rightarrow x \geq 0$$

$$\textcircled{4} |x| > x \Rightarrow x < 0$$

$$\text{** } \textcircled{5} |x| < x \Rightarrow x \in \phi$$

$$\textcircled{6} |-x| = |x|$$

$\forall \rightarrow$ for all.

$$|-x| = x ?$$

Suppose $x = -2$

$$\text{LHS} = | -(-2) | = |2| = 2$$

$$\text{RHS} = x = -2$$

$$\text{LHS} \neq \text{RHS}$$

$$\textcircled{7} |x^n| = |x|^n$$

$$\text{** } \textcircled{8} |xy| = |x||y|$$

$$\text{** } \textcircled{9} \left| \frac{x}{y} \right| = |x|/|y|$$

$$\text{V.Imp } \textcircled{10} \text{ (i) } ||x| - |y|| \leq |x + y| \leq |x| + |y|$$

$$\text{(ii) } ||x| - |y|| \leq |x - y| \leq |x| + |y|$$

$$\text{V.Imp } \textcircled{11} |x + y| = |x| + |y| \text{ for } xy \geq 0$$

$$\text{Ex } x = 0, y = -2 \Rightarrow xy = 0$$

$$\text{LHS} = |x + y| = |0 - 2| = 2$$

$$\text{RHS} = |x| + |y| = |0| + |-2| = 0 + 2 = 2 \} \text{LHS} = \text{RHS}$$

$$\textcircled{12} |x + y| < |x| + |y| \text{ for } xy < 0$$

$$\textcircled{13} |x| < a \Rightarrow -a < x < a$$

i.e $x \in (-a, a)$

$$|x| \leq a \Rightarrow -a \leq x \leq a$$

i.e $x \in [-a, a]$

$$|x| > a \Rightarrow x < -a \text{ (or) } x > a$$

i.e $x \in (-\infty, -a) \cup (a, \infty)$

$$|x| \geq a \Rightarrow x \leq -a \text{ (or) } x \geq a$$

i.e $x \in (-\infty, -a] \cup [a, \infty)$

less – lies

Ex 1) $|x| < 2 \Rightarrow -2 < x < 2$

i.e $x \in (-2, 2)$

2) $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2$

$$\Rightarrow -2 + 3 < x < 2 + 3$$

$$\Rightarrow 1 < x < 5$$

$$\Rightarrow x \in (1, 5)$$

3) $|2x - 3| \geq 2 \Rightarrow 2x - 3 \leq -2 \text{ (or) } 2x - 3 \geq 2$

$$\Rightarrow 2x \leq 1 \text{ (or) } 2x \geq 5$$

$$\Rightarrow x \leq \frac{1}{2} \Rightarrow x \geq \frac{5}{2}$$

$$\Rightarrow x \in (-\infty, \frac{1}{2}] \cup [\frac{5}{2}, \infty) = \mathbb{R} - (\frac{1}{2}, \frac{5}{2})$$

[diagram: number line from $-\infty$ to ∞ marking $\frac{1}{2}$ and $\frac{5}{2}$, shaded to the left of $\frac{1}{2}$ and to the right of $\frac{5}{2}$]

⑭ $|x| = k \Rightarrow x = \pm k.$

Ex $|x| = 2 \Rightarrow x = \pm 2$

$$\Rightarrow x \in \{-2, 2\}$$

⑮ $|x| = |y| \Rightarrow x = \pm y$

Ex $|x - 2| = |3 - 2x|$ then $x \in \{\frac{5}{3}, 1\}$

Sol $x - 2 = \pm(3 - 2x)$

$$x - 2 = 3 - 2x \text{ (or) } x - 2 = -(3 - 2x)$$

$$\Rightarrow x + 2x = 3 + 2$$

$$\Rightarrow 3x = 5$$

$$\Rightarrow x = \frac{5}{3}$$

for $x - 2 = -3 + 2x$

$$\Rightarrow x - 2x = -3 + 2$$

$$\Rightarrow -x = -1$$

$$\Rightarrow x = 1$$

Exs ① $|7| = 7$ ($7 \rightarrow +ve$)

② $|-7| = -(-7) = 7$ ($-7 \rightarrow -ve$)

③ $|0| = 0$

④ For $x \geq 3$, $|x - 3| = x - 3$ ($x - 3 \geq 0$)

For $x < 3$, $|x - 3| = -(x - 3) = 3 - x$ ($x - 3 < 0$)

⑤ $|2x - 3| = 2x - 3$ for $x \in \underline{\hspace{2cm}}$

Sol $2x - 3 \geq 0$

$$\Rightarrow 2x \geq 3$$

$$\Rightarrow x \geq \frac{3}{2}$$

$$\Rightarrow x \in \left[\frac{3}{2}, \infty\right)$$

⑥ $|2x - 3| = 3 - 2x$ for $x \in \underline{\hspace{2cm}}$

Sol $2x - 3 < 0$

$$\Rightarrow 2x < 3$$

$$\Rightarrow x < \frac{3}{2}$$

$$\Rightarrow x \in \left(-\infty, \frac{3}{2}\right)$$

⑦ Solve $|x + 2| = 5$

Sol $x + 2 = \pm 5$

$$\Rightarrow x + 2 = 5 \text{ (or) } x + 2 = -5$$

$$\Rightarrow x = 3 \text{ (or) } x = -7$$

$$\Rightarrow x \in \{-7, 3\}$$

$|x| = k$

$$\Rightarrow x = \pm k$$

⑧ $|2x - 3| = |x^2 + 2x|$ then $x = -2 \pm \sqrt{7}$

Sol $2x - 3 = \pm(x^2 + 2x)$

$$2x - 3 = x^2 + 2x \text{ (or) } 2x - 3 = -(x^2 + 2x)$$

$$\Rightarrow x^2 = -3 \text{ (no real } x)$$

$$\Rightarrow 2x - 3 = -x^2 - 2x$$

$$\Rightarrow x^2 + 4x - 3 = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{16 + 12}}{2}$$

$$= \frac{-4 \pm \sqrt{28}}{2} = \frac{-4 \pm 2\sqrt{7}}{2} = -2 \pm \sqrt{7}$$

$|x| = |y|$

$$\Rightarrow x = \pm y$$

$$a = 1$$

$$b = 4$$

$$c = -3$$

⑨ $|x - 2| < 3$ then $x \in \underline{\hspace{2cm}}$

Sol $-3 < x - 2 < 3$

$$\Rightarrow -3 + 2 < x < 3 + 2$$

$$\Rightarrow -1 < x < 5$$

$$\Rightarrow x \in (-1, 5)$$

⑩ Solve $|2x - 3| \leq 3$

Sol $-3 \leq 2x - 3 \leq 3$

$$\Rightarrow -3 + 3 \leq 2x \leq 3 + 3$$

$$\Rightarrow 0 \leq 2x \leq 6$$

$$\Rightarrow 0 \leq x \leq 3 \Rightarrow x \in [0, 3]$$

$$|x| < a$$

$$\Rightarrow -a < x < a$$

$$|x| \leq a$$

$$\Rightarrow -a \leq x \leq a.$$

H.W Solve

① $|x - 2| > 5$

② $|3 - x| \geq 7$

③ $|x - 2| \geq 5$

④ $|x^2 + 2x + 5| = 7$

① Sol $|x - 2| > 5$

$$\Rightarrow x - 2 < -5 \text{ (or) } x - 2 > 5$$

$$\Rightarrow x < -5 + 2 \text{ (or) } x > 5 + 2$$

$$\Rightarrow x < -3 \text{ or } x > 7$$

$$\Rightarrow x \in (-\infty, -3) \cup (7, \infty)$$

[diagram: number line from $-\infty$ to ∞ with open circles at -3 and 7 , shaded outward]

$$|x| > a$$

$$\Rightarrow x < -a \text{ (or) } x > a$$

$$\textcircled{2} |3 - x| \geq 7$$

$$\Rightarrow 3 - x \leq -7 \text{ (or) } 3 - x \geq 7$$

$$\Rightarrow -x \leq -10 \mid -x \geq 4$$

$$\Rightarrow x \geq 10 \mid x \leq -4$$

[diagram: number line from $-\infty$ to ∞ marking -4 and 10 , shaded left of -4 and right of 10 with closed dots]

$$x \in (-\infty, -4] \cup [10, \infty)$$

$$\textcircled{4} |x^2 + 2x + 5| = 7$$

$$\Rightarrow x^2 + 2x + 5 = \pm 7$$

$$\Rightarrow x^2 + 2x + 5 = 7 \text{ (or) } x^2 + 2x + 5 = -7$$

$$\Rightarrow x^2 + 2x - 2 = 0 \text{ (or) } x^2 + 2x + 12 = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 + 8}}{2}$$

$$= \frac{-2 \pm \sqrt{12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}}{2}$$

$$= -1 \pm \sqrt{3}$$

Here $\Delta < 0$

No real roots.

$$|x| = k$$

$$\Rightarrow x = \pm k$$

→ Solve $|x + 3| + |2 - 3x| = 5$ —①

Sol $x + 3 = 0 \Rightarrow x = -3 \mid 2 - 3x = 0 \Rightarrow x = \frac{2}{3}$

Critical points = $-3, \frac{2}{3}$

[diagram: number line from $-\infty$ to ∞ marking -3 and $\frac{2}{3}$]

Case (i) Suppose $x < -3$

$$\textcircled{1} \Rightarrow |x + 3| + |2 - 3x| = 5 \text{ (} x + 3 < 0, 2 - 3x > 0 \text{)}$$

$$\Rightarrow -x - 3 + 2 - 3x = 5$$

$$\Rightarrow -4x = 6$$

$$\Rightarrow x = -\frac{3}{2} > -3$$

No real x.

Case (ii) Suppose $-3 \leq x \leq \frac{2}{3}$

$$\textcircled{1} \Rightarrow |x + 3| + |2 - 3x| = 5 \quad (x + 3 \geq 0, 2 - 3x \geq 0)$$

$$\Rightarrow x + 3 + 2 - 3x = 5$$

$$\Rightarrow -2x = 0$$

$$\Rightarrow x = 0$$

$\therefore x = 0$ is the solution

Case (iii) Suppose $x > \frac{2}{3}$

$$\textcircled{1} \Rightarrow |x + 3| + |2 - 3x| = 5 \quad (x + 3 > 0, 2 - 3x < 0)$$

$$\Rightarrow x + 3 + 3x - 2 = 5$$

$$\Rightarrow 4x = 4$$

$$\Rightarrow x = 1$$

$\therefore x = 1$ is the sol.

Ans $x \in \{0, 1\}$

$$x \leq \frac{2}{3}$$

$$3x \leq 2$$

$$2 - 3x \geq 0$$

$$-3 \leq x, 0 \leq x + 3$$

$$x > \frac{2}{3}$$

$$3x \leq 2, 2 - 3x \geq 0$$

$$3x > 2, 2 - 3x < 0$$

→ **Solve** $|3x - 2| + x = 11$

Sol Critical point = $\frac{2}{3}$

$$3x - 2 = 0$$

$$x = \frac{2}{3}$$

[diagram: number line from $-\infty$ to ∞ marking $\frac{2}{3}$]

Case (i) Suppose $x < \frac{2}{3}$

$$|3x - 2| + x = 11 \quad (3x - 2 < 0)$$

$$\Rightarrow 2 - 3x + x = 11$$

$$\Rightarrow -2x = 9$$

$$\Rightarrow x = -\frac{9}{2} \text{ is the solution.}$$

Case (ii) Suppose $x \geq \frac{2}{3}$

$$|3x - 2| + x = 11 \quad (3x - 2 \geq 0)$$

$$\Rightarrow 3x - 2 + x = 11$$

$$\Rightarrow 4x = 13$$

$$\Rightarrow x = \frac{13}{4} \text{ is also a solution}$$

$$\frac{2}{3} < \frac{13}{4} \quad (8, 39)$$

Ans $x \in \{-\frac{9}{2}, \frac{13}{4}\}$

$$x < \frac{2}{3}$$

$$3x < 2$$

$$3x - 2 < 0$$

→ Solve $|x + 2| - |x - 1| < x - \frac{3}{2}$

Sol Critical pts = -2, 1

[diagram: number line from $-\infty$ to ∞ marking -2 and 1]

Case (i) suppose $x < -2$

$$|x + 2| - |x - 1| < x - \frac{3}{2} \quad (x + 2 < 0, x - 1 < 0)$$

$$\Rightarrow (-x - 2) - (1 - x) < x - \frac{3}{2}$$

$$\Rightarrow -x - 2 - 1 + x < x - \frac{3}{2}$$

$$\Rightarrow -3 < x - \frac{3}{2}$$

$$\Rightarrow -3 + \frac{3}{2} < x$$

$$\Rightarrow -\frac{3}{2} < x$$

$$\Rightarrow x > -\frac{3}{2}$$

[diagram: number line marking -2 and $-\frac{3}{2}$]

No solution

Case (ii) suppose $-2 \leq x \leq 1$

$$|x + 2| - |x - 1| < x - \frac{3}{2} \quad (x + 2 \geq 0, x - 1 < 0)$$

$$\Rightarrow (x + 2) - (1 - x) < x - \frac{3}{2}$$

$$\Rightarrow x + 2 - 1 + x < x - \frac{3}{2}$$

$$\Rightarrow x < -\frac{3}{2} - 1$$

$$\Rightarrow x < -\frac{5}{2}$$

[diagram: number line marking $-\frac{5}{2}$, -2 , 1]

No solution

Case (iii) suppose $x > 1$

$$|x + 2| - |x - 1| < x - \frac{3}{2} \quad (x + 2 > 0, x - 1 > 0)$$

$$\Rightarrow (x + 2) - (x - 1) < x - \frac{3}{2}$$

$$\Rightarrow x + 2 - x + 1 < x - \frac{3}{2}$$

$$\Rightarrow 3 < x - \frac{3}{2}$$

$$\Rightarrow \frac{9}{2} < x$$

$$\Rightarrow x > \frac{9}{2}$$

[diagram: number line marking 1 and $\frac{9}{2}$]

$$x > \frac{9}{2}$$

Ans $(\frac{9}{2}, \infty)$

$$x + 2 = 0$$

$$x = -2$$

$$x - 1 = 0$$

$$\Rightarrow x = 1$$

Ex Solve $x^2 - 5|x| + 6 < 0$

Sol: $x = 0 \rightarrow$ Critical point

[diagram: number line from $-\infty$ to ∞ marking 0]

Case (i) suppose $x < 0$

$$x^2 - 5|x| + 6 < 0$$

$$\Rightarrow x^2 + 5x + 6 < 0$$

$$\Rightarrow (x + 2)(x + 3) < 0$$

$$\Rightarrow x \in (-3, -2)$$

Common values $(-3, -2)$

Case (ii) suppose $x \geq 0$

$$x^2 - 5|x| + 6 < 0$$

$$x^2 - 5x + 6 < 0$$

$$\Rightarrow (x - 2)(x - 3) < 0$$

$$\Rightarrow x \in (2, 3)$$

Common values (2, 3)

From (i), (ii)

$$x \in (-3, -2) \cup (2, 3)$$

$$|x - 1|$$

Ex solve $|x^2 - 4x| < 5$

Sol $\Rightarrow -5 < x^2 - 4x < 5$

$$\Rightarrow x^2 - 4x + 5 > 0 \text{ and } x^2 - 4x - 5 < 0$$

$$a > 0, \Delta < 0$$

[graph: upward parabola above x-axis]

$$\Rightarrow x \in \mathbb{R}.$$

$$a > 0, \Delta = 16 + 20 = 36 > 0$$

$$x = \frac{4 \pm 6}{2} = -1, 5$$

[graph: upward parabola cutting x-axis at -1 and 5, - between]

$$\Rightarrow x \in (-1, 5)$$

common values (-1, 5)

$$|x| < a$$

$$\Rightarrow -a < x < a$$

*** Ex Solve $|x^2 - 2x - 8| + |x^2 + x - 2| = 3|x + 2|$

Sol $|x^2 - 2x - 8| + |x^2 + x - 2|$

$$= |(x^2 - 2x - 8) - (x^2 + x - 2)|$$

$$= |-3x - 6|$$

$$= |3x + 6|$$

$$= 3|x + 2|$$

This occurs when

$$(x^2 - 2x - 8)(x^2 + x - 2) \leq 0$$

$$\Rightarrow (x - 4)(x + 2)(x + 2)(x - 1) \leq 0$$

$$|x + y| = |x| + |y| \text{ if } xy \geq 0 \quad \checkmark$$

$$|x - y| = |x| + |y| \text{ if } xy \leq 0$$

[diagram: sign chart on number line from $-\infty$ to ∞ , critical points -2, 1, 4 (each = 0); signs +, +, -, +]

$$\Rightarrow x \in [1, 4] \cup \{-2\}$$

→ The number of integers satisfying the equation

$$|x| + \left| \frac{4 - x^2}{x} \right| = \left| \frac{4}{x} \right| \text{ is } 5$$

Sol → $|x| + |4 - x^2|/|x| = 4/|x|$ where $|x| \neq 0$

$$\Rightarrow |x|^2 + |4 - x^2| = 4$$

$$\Rightarrow x^2 + |x^2 - 4| = 4$$

$$\Rightarrow x^2 + |(x + 2)(x - 2)| = 4 \text{ --- ①}$$

[diagram: number line from $-\infty$ to ∞ marking -2 and 2]

Case (i) suppose $x \in (-\infty, -2] \cup [2, \infty)$

$$\text{①} \Rightarrow x^2 + (x + 2)(x - 2) = 4$$

$$\Rightarrow x^2 + x^2 - 4 = 4$$

$$\Rightarrow 2x^2 = 8$$

$$\therefore \left| \frac{x}{y} \right| = |x|/|y|$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

Ans $\{-2, 2\}$

Case (ii) suppose $x \in (-2, 2)$

$$\text{①} \Rightarrow x^2 - (x + 2)(x - 2) = 4$$

$$\Rightarrow x^2 - (x^2 - 4) = 4$$

$$\Rightarrow 4 = 4$$

$$\therefore x \in (-2, 2)$$

From (i), (ii)

$x \in [-2, 2] \rightarrow$ No. of integral values = 5

$-2, -1, 0, 1, 2$

→ The no. of integral values of x satisfying

$$|x - |x - 4|| = 4 \text{ is } \infty$$

Sol → $x - |x - 4| = \pm 4$

$$\Rightarrow x - |x - 4| = 4 \text{ (or) } x - |x - 4| = -4$$

$$\Rightarrow x - 4 = |x - 4|$$

$$\Rightarrow |x - 4| = x - 4$$

$$\Rightarrow x - 4 \geq 0$$

$$\Rightarrow x \geq 4$$

$$\Rightarrow x \in [4, \infty)$$

$$(|T| = T, T \geq 0)$$

$$x + 4 = |x - 4|$$

$$\Rightarrow |x - 4| = x + 4 \text{ --- ①}$$

[diagram: number line from $-\infty$ to ∞ marking 4]

Suppose $x < 4$

$$\textcircled{1} \Rightarrow 4 - x = x + 4$$

$$\Rightarrow 0 = 2x$$

$$\Rightarrow x = 0$$

(or) suppose $x > 4$

$$\textcircled{1} \Rightarrow x - 4 = x + 4$$

$$\Rightarrow -4 = 4 \text{ X}$$

No sol.

Finally $x \in [4, \infty) \cup \{0\}$

$$|x| = k$$

$$\Rightarrow x = \pm k$$

→ Solve $|x + \frac{1}{x}| > 2$

Sol $x + \frac{1}{x} < -2$ (or) $x + \frac{1}{x} > 2$

$$\Rightarrow \frac{x^2 + 1}{x} < -2$$

$$\Rightarrow \frac{x^2 + 1}{x} + 2 < 0$$

$$\Rightarrow \frac{x^2 + 1 + 2x}{x} < 0$$

$$\Rightarrow \frac{(x + 1)^2}{x} < 0$$

$$\Rightarrow x < 0 \text{ --- ①}$$

$$\Rightarrow \frac{x^2 + 1}{x} > 2$$

$$\Rightarrow \frac{x^2 + 1}{x} - 2 > 0$$

$$\Rightarrow \frac{x^2 + 1 - 2x}{x} > 0$$

$$\Rightarrow \frac{x^2 - 2x + 1}{x} > 0$$

$$\Rightarrow \frac{(x - 1)^2}{x} > 0$$

$$\Rightarrow x > 0 \text{ --- ②}$$

$$|x| > a$$

$$\Rightarrow x < -a \text{ (or) } x > a$$

From ①, ②

$$x \in \mathbb{R} - \{0\}$$

[diagram: number line from $-\infty$ to ∞ with open circle at 0]

→ Number of integers satisfying $|x - 1|/(x + 2) > 1$ is

Sol Given $|x - 1|/(x + 2) > 1$ —①

[diagram: number line from $-\infty$ to ∞ marking 1]

Case (i) $x < 1$

$$\Rightarrow x - 1 < 0$$

$$\textcircled{1} \Rightarrow \frac{1 - x}{x + 2} > 1$$

$$\Rightarrow \frac{1 - x}{x + 2} - 1 > 0$$

$$\Rightarrow \frac{1 - x - x - 2}{x + 2} > 0$$

$$\Rightarrow \frac{-1 - 2x}{x + 2} > 0$$

$$\Rightarrow \frac{1 + 2x}{x + 2} < 0$$

$$\Rightarrow (1 + 2x)(x + 2) < 0$$

$$\Rightarrow x \in (-2, -\frac{1}{2})$$

[diagram: number line marking $-2, -1, -\frac{1}{2}, 0$ with open interval from -2 to $-\frac{1}{2}$]

Integral value of $x = -1$

$$\frac{a}{b} < 0$$

$$\Rightarrow ab < 0 \checkmark$$

[diagram: sign chart with + left of -2 , - between -2 and $-\frac{1}{2}$, + right]

Case (ii) $x \geq 1 \Rightarrow x - 1 \geq 0$

$$\textcircled{1} \Rightarrow \frac{x - 1}{x + 2} > 1$$

$$\Rightarrow \frac{x - 1}{x + 2} - 1 > 0$$

$$\Rightarrow \frac{x - 1 - x - 2}{x + 2} > 0$$

$$\Rightarrow -\frac{3}{x + 2} > 0$$

$$\Rightarrow \frac{3}{x + 2} < 0$$

$$\Rightarrow x + 2 < 0$$

$$\Rightarrow x < -2$$

→ No Common value

From (i), (ii)

The integral value of x is -1

No. of integers

$$= 1$$

Ex Solve $\left| \frac{12x}{4x^2 + 9} \right| \leq 1$

Sol $\Rightarrow 12|x|/(4x^2 + 9) \leq 1$ —①

[diagram: number line from $-\infty$ to ∞ marking 0]

(i) Suppose $x < 0$

$$\textcircled{1} \Rightarrow -\frac{12x}{4x^2 + 9} \leq 1$$

$$\Rightarrow -12x \leq 4x^2 + 9$$

$$\Rightarrow 4x^2 + 12x + 9 \geq 0$$

$$\Rightarrow (2x + 3)^2 \geq 0$$

$$\Rightarrow x \in \mathbb{R}$$

→ Common values are $x < 0$

$$\Rightarrow x \in (-\infty, 0)$$

Case (ii) Suppose $x \geq 0$

$$\textcircled{1} \Rightarrow \frac{12x}{4x^2 + 9} \leq 1$$

$$\Rightarrow 12x \leq 4x^2 + 9$$

$$\Rightarrow 4x^2 - 12x + 9 \geq 0$$

$$\Rightarrow (2x - 3)^2 \geq 0$$

$$\Rightarrow x \in \mathbb{R}$$

Common values are $x \geq 0$

$$\Rightarrow x \in [0, \infty)$$

From (i), (ii)

$$x \in (-\infty, 0) \cup [0, \infty)$$

$$\Rightarrow x \in \mathbb{R}.$$

Ex Solve $|x - 1| + |x - 2| = 0$ —①

Sol

① occurs if $|x - 1| = 0$ and $|x - 2| = 0$

There is no x satisfying

Ans No real x

Square root of real number

Consider $a^2 = b \Rightarrow \sqrt{b} = a$

Ex $2^2 = 4 \Rightarrow \sqrt{4} = 2$

$3^2 = 9 \Rightarrow \sqrt{9} = 3$

Square root function

let $y = f(x) = \sqrt{x}$ is square function

Graph of $y = f(x) = \sqrt{x}$

Domain = $[0, \infty)$, Range = $[0, \infty)$

[graph: curve starting at origin rising slowly to the right, through (1, 1) and about (4, 2); x-axis marked 1 to 4, y-axis marked 1, 2]

Note $y = f(x) = \sqrt{x}$ where $x \geq 0$

Ex $\sqrt{x - 2}$ is defined when $x - 2 \geq 0$

$$\Rightarrow x \geq 2$$

$$\Rightarrow x \in [2, \infty)$$

$$\therefore \text{Domain of } \sqrt{x - 2} \text{ is } [2, \infty)$$

Ex Domain of $\sqrt{2x - 3}$ is $[\frac{3}{2}, \infty)$

Sol $2x - 3 \geq 0$

$$\Rightarrow x \geq \frac{3}{2}$$

$$\Rightarrow x \in [\frac{3}{2}, \infty)$$

Ex Domain of $\sqrt{x^2 - 5x + 6}$ is ____

Sol $x^2 - 5x + 6 \geq 0$

$$\Rightarrow (x - 2)(x - 3) \geq 0$$

$$\Rightarrow x \in (-\infty, 2] \cup [3, \infty)$$

[graph: upward parabola cutting x-axis at 2 and 3 (each = 0); + outside, - between]

Ex Domain of $\sqrt{x - |x|}$ is

Sol $x - |x| \geq 0$

$$\Rightarrow x \geq |x|$$

$$\Rightarrow |x| \leq x$$

$$\Rightarrow |x| = x$$

$$\Rightarrow x \geq 0$$

$$\Rightarrow x \in [0, \infty)$$

$$\therefore \text{Domain} = [0, \infty)$$

$$\because |x| < x \text{ No real } x$$

Note $y = f(x) = \frac{1}{\sqrt{x}}$ where $x > 0$

Ex Domain of $\frac{1}{\sqrt{x-2}}$ is

Sol $x - 2 > 0$

$$\Rightarrow x > 2$$

$$\Rightarrow x \in (2, \infty)$$

$$\text{Domain} = (2, \infty)$$

$$\sqrt{x-2}$$

Step of a real number

If $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ and $n \leq x < n + 1$

then $[x] = n$

(Read it as step x)

Ex $0 \leq x < 1 \Rightarrow [x] = 0$

$1 \leq x < 2 \Rightarrow [x] = 1$

$2 \leq x < 3 \Rightarrow [x] = 2$

$-1 \leq x < 0 \Rightarrow [x] = -1$

$-2 \leq x < -1 \Rightarrow [x] = -2$

Exs: $[0] = 0$

$$[0.1] = 0$$

$$[0.5] = 0$$

$$[0.9] = 0$$

$$[0.9999] = 0$$

$$[1] = 1$$

$$[1.1] = 1$$

$$[1.5] = 1$$

$$[1.9] = 1$$

$$[1.9999] = 1$$

$$[2] = 2$$

$$[2.1] = 2$$

$$[2.5] = 2$$

$$[2.9] = 2$$

$$[2.999] = 2$$

$$[-0.1] = -1$$

[diagram: number line marking -1 , -0.1 , 0 , 1 with arrow from -0.1 back to -1]

$$[-0.2] = -1$$

$$[-0.9] = -1$$

$$[-1] = -1$$

$$[-1.2] = -2$$

$$[-1.7] = -2$$

$$[-1.999] = -2$$

$$[-2] = -2$$

[diagram: number line marking -2 , -1.2 , -1]

[diagram: number line from -3 to 3 with braces over $[-1, 0)$, $[0, 1)$, $[1, 2)$ each with an arrow back to its left endpoint; 1.2 marked]

Exs: $[0.5] = 0$

$$[-0.5] = -1$$

$$[1.2] = 1$$

$$[-1.2] = -2$$

$$[2.7] = 2$$

$$[-2.7] = -3$$

$$[0] = 0$$

$$[1] = 1$$

$$[2] = 2$$

$$[-1] = -1$$

$$[-2] = -2$$

Def Step function (or) Greatest integer function

(or) floor function

let $y = f(x) = [x]$ is called a step function

Domain = $\mathbb{R}$, Range = $\mathbb{Z}$

Graph of $f(x) = [x]$

[graph: step graph with horizontal segments: $y = -2$ on $[-2, -1)$, $y = -1$ on $[-1, 0)$, $y = 0$ on $[0, 1)$, $y = 1$ on $[1, 2)$, $y = 2$ on $[2, 3)$; closed dot at left end, open circle at right end of each]

properties

① $[x] \leq x, \forall x \in \mathbb{R}$

② $[x] = x, \forall x \in \mathbb{Z}$

③ $[x] < x, \forall x \in \mathbb{R} - \mathbb{Z}$.

④ $[x] > x$, for no real x

*** ⑤ $\forall x \in \mathbb{R}, k \in \mathbb{Z}$ then $[x + k] = [x] + k$

Ex $[x + 1] = [x] + 1$

$[x + \frac{1}{2}] \neq [x] + \frac{1}{2}$

⑥ $[-x] = -[x], \forall x \in \mathbb{Z}$

⑦ $[-x] = -[x] - 1, \forall x \in \mathbb{R} - \mathbb{Z}$

Ex $x = 1.3$

LHS = $[-x] = [-1.3] = -2$

RHS = $-[x] - 1$

$= -[1.3] - 1$

$= -1 - 1$

$= -2$

$\therefore \text{LHS} = \text{RHS}$

*** ⑧ $[x] + [-x] = 0, \forall x \in \mathbb{Z}$

$[x] + [-x] = -1, \forall x \in \mathbb{R} - \mathbb{Z}$

⑨ $[[x]] = [x], \forall x \in \mathbb{R}$

*** ⑩ $x - 1 < [x] \leq x, \forall x \in \mathbb{R}$

Exs: 1) $[x] < 2$ then $x \in (-\infty, 2)$

[diagram: number line from $-\infty$ to ∞ marking 0 to 4, with brace loops over $[0, 1)$, $[1, 2)$, $[2, 3)$]

② $[x] \leq 2$ then $x \in (-\infty, 3)$

[diagram: number line from $-\infty$ to ∞ marking 0 to 4, with brace loop over $[2, 3)$]

③ $[x] > 2$ then $x \in [3, \infty)$.

[diagram: number line from $-\infty$ to ∞ marking 0 to 4, with brace loops over $[2, 3)$ and $[3, 4)$]

④ $[x] \geq 2$ then $x \in [2, \infty)$

[diagram: number line from $-\infty$ to ∞ marking 0 to 4, with brace loop starting at 2]

⑤ $[x] > -3$ then $x \in [-2, \infty)$

[diagram: number line from $-\infty$ to ∞ marking -5 to -1, with brace loops over $[-3, -2)$ and $[-2, -1)$]

⑥ $[x] \leq -5$ then $x \in (-\infty, -4)$

[diagram: number line from $-\infty$ to ∞ marking -7 to -3, with brace loop over $[-5, -4)$]

Ans $[-5, \infty)$ ✗

$(-\infty, -4)$ ✓

$(-\infty, -5]$ ✗

H.W

① $[x] < 7$ then $x \in \underline{\hspace{2cm}}$

② $[x] \geq -6$ then $x \in \underline{\hspace{2cm}}$

③ $[x] > -4$ then $x \in \underline{\hspace{2cm}}$.

Ex: Domain of $\sqrt{x - [x]}$ is $\underline{\hspace{2cm}}$

Sol $x - [x] \geq 0$

$$\Rightarrow x \geq [x]$$

$$\Rightarrow [x] \leq x$$

$$\Rightarrow x \in \mathbb{R}$$

Ex: Domain of $\frac{1}{\sqrt{[x] - x}}$ is

Sol $[x] - x > 0$

$$\Rightarrow [x] > x$$

No real x

$x \in \phi$

fractional part of x

$$1.7 = 1 + 0.7 \text{ (1} \rightarrow \text{integral part, 0.7} \rightarrow \text{fractional part)}$$

$$-1.7 = -1 - 0.7$$

$$= -1 - 1 + 1 - 0.7$$

$$= -2 + 0.3 \text{ (-2} \rightarrow \text{integral part, 0.3} \rightarrow \text{fractional part)}$$

$$2 = 2 + 0 \text{ (0} \rightarrow \text{fractional part)}$$

Def If $x \in \mathbb{R}$, the function $f(x) = x - [x]$

is called fractional part of x , it is

denoted by $]x[$ (or) $\{x\}$

$$\therefore \{x\} = x - [x]$$

$$\text{Ex } \{1.7\} = 1.7 - [1.7] = 1.7 - 1 = 0.7$$

$$\{-1.7\} = -1.7 - [-1.7] = -1.7 - (-2) = -1.7 + 2 \\ = 0.3$$

$$\{2.3\} = 0.3$$

$$\{-2.3\} = 0.7$$

$$\{2\} = 0$$

$$-1.7 = -2 + 0.3 \text{ (-2} \rightarrow \text{Integral, 0.3} \rightarrow \text{fractional)}$$

$$-2.3 = -3 + 0.7$$

Note Domain of $x - [x]$ is $\mathbb{R}$

Range of $x - [x]$ is $[0, 1)$

Graph of $x - [x]$:

$$x = 0, y = x - [x] = 0$$

$$x = 0.3, y = 0.3$$

$$x = 0.7, y = 0.7$$

$$x = 0.9, y = 0.9$$

$$x = 0.9999, y = 0.9999$$

$$x = 1, y = 0$$

$$x = 1.3, y = 0.3$$

$$x = 1.7, y = 0.7$$

$$x = 1.9, y = 0.9$$

$$x = 1.9999, y = 0.9999$$

[graph: sawtooth graph — line segments rising from each integer $-2, -1, 0, 1, 2$ on the x -axis up towards height 1, open circle at the top of each]

period of $x - [x] = 1$

Ex Range of $[x] - x$ is ____

Sol Range of $x - [x]$ is $[0, 1)$

Range of $[x] - x$ is $(-1, 0]$

$$0 \leq x - [x] < 1$$

$$0 \geq [x] - x > -1$$

$$-1 < [x] - x \leq 0$$

Exponential function

If $a > 0$, $x \in \mathbb{R}$ then $f(x) = a^x$ is called exponential function.

Ex: $2^x, \left(\frac{1}{2}\right)^x = 2^{-x}$

Graph of 2^x

[graph: increasing curve approaching x-axis on the left, passing through (0, 1), rising steeply on the right]

Graph of $\left(\frac{1}{2}\right)^x = 2^{-x}$

[graph: decreasing curve, steep on the left, through a point on the y-axis, approaching x-axis on the right]

Note Domain of a^x ($a > 0$) is $\mathbb{R}$.

Range of a^x is $(0, \infty)$

Graph of $f(x) = \frac{1}{x}$

[graph: rectangular hyperbola — one branch in the first quadrant, one in the third quadrant, approaching both axes]

Domain of $\frac{1}{x} = \mathbb{R} - \{0\}$

Range of $\frac{1}{x} = \mathbb{R} - \{0\}$

Ex $\forall x \in [-3, 5] - \{0\}$ then $\frac{1}{x} \in$ ____.

Sol [graph: hyperbola $y = \frac{1}{x}$ with x-axis marked -3, 0, 5; values $\frac{1}{5}$ and $-\frac{1}{3}$ marked on the y-axis with arrows upward from $\frac{1}{5}$ and downward from $-\frac{1}{3}$]

$$\frac{1}{x} \in (-\infty, -\frac{1}{3}] \cup [\frac{1}{5}, \infty)$$

Graph of $f(x) = \frac{1}{x^2}$

[graph: two branches above the x-axis, rising steeply near the y-axis on both sides and approaching the x-axis outward]

Domain = $\mathbb{R} - \{0\}$

Range = $(0, \infty)$

Graph of $f(x) = x^2$

[graph: upward parabola with vertex at $(0, 0)$]

Domain = $\mathbb{R}$

Range = $[0, \infty)$

Graph of $f(x) = x^3$

[graph: cubic curve through the origin, rising to the right and falling to the left; axes marked $-1, 1, 2$ on x and $-1, 1$ on y]

Domain = $\mathbb{R}$

Range = $(-\infty, \infty) = \mathbb{R}$.

Graph of $x + \frac{1}{x}$.

[graph: branch in first quadrant with minimum at $(1, 2)$; branch in third quadrant with maximum at $(-1, -2)$; $-\frac{1}{2}$ marked on x-axis]

Domain = $\mathbb{R} - \{0\}$

range = $(-\infty, -2] \cup [2, \infty)$

Function | Domain | Range

① $\log_a x$ | $(0, \infty)$ | $\mathbb{R}$

② a^x | $(-\infty, \infty) = \mathbb{R}$ | $(0, \infty)$

③ $[x]$ | $\mathbb{R}$ | $\mathbb{Z}$

④ $x - [x]$ | $\mathbb{R}$ | $[0, 1)$

⑤ $|x|$ | $\mathbb{R}$ | $[0, \infty)$

⑥ $\sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$

⑦ $x + \frac{1}{x}$ | $\mathbb{R} - \{0\}$ | $(-\infty, -2] \cup [2, \infty)$

[sketches: small log curve and small exponential curve beside rows ① and ②]

Graph of $\log_a x$

(i) $a > 1$

[graph: increasing curve through (1, 0), dropping steeply near the y-axis]

(ii) $0 < a < 1$

[graph: decreasing curve through (1, 0), rising steeply near the y-axis]

Graph of a^x

(i) $a > 1$

[graph: increasing curve approaching x-axis on the left, crossing y-axis at 1, rising steeply on the right]

(ii) $0 < a < 1$

[graph: decreasing curve, steep on the left, crossing y-axis at 1, approaching x-axis on the right]

Graph of $[x]$

[graph: step graph with horizontal segments $y = -2$ on $[-2, -1)$, $y = -1$ on $[-1, 0)$, $y = 0$ on $[0, 1)$, $y = 1$ on $[1, 2)$; open circles at right ends; axes marked -2 to 2 on x , -2 to 1 on y]

Graph of $x - [x]$

[graph: sawtooth graph — line segments rising from each integer $-2, -1, 0, 1, 2$ towards height 1, open circle at the top of each; x-axis marked -2 to 3]

Graph of $\sqrt{x}$

[graph: curve starting at origin, rising through (1, 1) and flattening to the right; x-axis marked 1, 2]

$$\sqrt{2} = 1.414$$

Graph of $x + \frac{1}{x}$

[graph: branch in first quadrant with minimum at (1, 2); branch in third quadrant with maximum at $(-1, -2)$; x-axis marked $-2, -1, 1, 2$]