

RUNNING NOTES

Trigonometry up to Transformations

Teacher's handwritten notes, typed

IN THESE NOTES

1. Trigonometric Ratio's

Sign's of Trigonometric ratio's

Domain and ranges of Trigonometric functions

Ratio and proportion

Compound angles

Multiple and Sub multiple angles

componendo and dividendo Rule

Transformations

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1. Trigonometric Ratio's

→ [diagram: axes OX, OY ; point $P(x, y)$ joined to O with length r , angle θ at O ; horizontal x and vertical dotted line y from P]

$$\sin\theta = \frac{y}{r} \mid \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \mid \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \mid \cot\theta = \frac{x}{y}$$

If θ is acute ($0^\circ < \theta < 90^\circ$)

$$\text{then } \sin\theta = \frac{y}{r} = \frac{\text{opp.side of } \theta}{\text{Hypo.}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{Adjacent side of } \theta}{\text{Hypo.}}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{opp}}{\text{Adj}}$$

$$\rightarrow \csc\theta = \frac{1}{\sin\theta} \Rightarrow \sin\theta \cdot \csc\theta = 1$$

$$\sec\theta = \frac{1}{\cos\theta} \Rightarrow \cos\theta \cdot \sec\theta = 1$$

$$\cot\theta = \frac{1}{\tan\theta} \Rightarrow \tan\theta \cdot \cot\theta = 1$$

$$\rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\rightarrow \sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\rightarrow \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \sec^2\theta = 1 + \tan^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$\rightarrow \csc^2\theta - \cot^2\theta = 1$$

$$\Rightarrow \csc^2\theta = 1 + \cot^2\theta$$

$$\cot^2\theta = \csc^2\theta - 1$$

→ [diagram: circle with a full-rotation arrow around the centre and a radius]

$$1^\circ = \frac{1}{360} \text{ (complete rotation (or) angle)}$$

$$1^g = \frac{1}{400} \text{ (complete angle)}$$

→ [diagram: circle, sector with two radii r and arc of length r , angle 1^c at centre]

$$1^c = 1 \text{ radian}$$

$$\text{Complete angle} = 360^\circ = 400^g = 2\pi^c$$

Note : $180^\circ = 200^g = \pi$

$$r \text{ — } 1^c$$

$$2\pi r \rightarrow 2\pi^c$$

→ $360^\circ = 2\pi$

$$180^\circ = \pi$$

$$90^\circ = \frac{\pi}{2}$$

$$60^\circ = \frac{\pi}{3}$$

$$45^\circ = \frac{\pi}{4}$$

$$30^\circ = \frac{\pi}{6}$$

$$0^\circ = 0$$

30° family members

$$30^\circ = \frac{\pi}{6}$$

$$150^\circ = 180^\circ - 30^\circ = \frac{5\pi}{6}$$

$$210^\circ = 180^\circ + 30^\circ = \frac{7\pi}{6}$$

$$330^\circ = 360^\circ - 30^\circ = \frac{11\pi}{6}$$

$$45^\circ = \frac{\pi}{4}$$

$$135^\circ = 180^\circ - 45^\circ = \frac{3\pi}{4}$$

$$225^\circ = 180^\circ + 45^\circ = \frac{5\pi}{4}$$

$$315^\circ = 360^\circ - 45^\circ = \frac{7\pi}{4}$$

$$60^\circ = \frac{\pi}{3}$$

$$120^\circ = 180^\circ - 60^\circ = \frac{2\pi}{3}$$

$$240^\circ = 180^\circ + 60^\circ = \frac{4\pi}{3}$$

$$300^\circ = 360^\circ - 60^\circ = \frac{5\pi}{3}$$

$$\rightarrow 1 \times 360^\circ = 360^\circ = 2\pi \checkmark$$

$$2 \times 360^\circ = 720^\circ = 4\pi \checkmark$$

$$3 \times 360^\circ = 1080^\circ = 6\pi \checkmark$$

$$4 \times 360^\circ = 1440^\circ = 8\pi \checkmark$$

$$5 \times 360^\circ = 1800^\circ = 10\pi. \checkmark$$

→ Sign's of Trigonometric ratio's

[diagram: axes $X'OX$ and $Y'OY$ dividing the plane into four quadrants]

$II_Q (90^\circ < \theta < 180^\circ)$

$\sin, \csc \}$ +ve.

Remaining T.R's are -ve.

$I_Q (0^\circ < \theta < 90^\circ)$

All T.R's are +ve.

$III_Q (180^\circ < \theta < 270^\circ)$

$\tan, \cot \}$ +ve

$IV_Q (270^\circ < \theta < 360^\circ)$

$\cos, \sec \}$ +ve.

Reason

[diagram: axes with origin O ; point $P(x, y)$ in second quadrant joined to O by r ; angle θ at O measured from positive x -axis]

$$90^\circ < \theta < 180^\circ$$

$$\Rightarrow P(x, y) \in II_Q$$

$$\Rightarrow x < 0, y > 0$$

r , always +ve.

$$\checkmark \sin \theta = \frac{y}{r} > 0 \mid \csc \theta > 0$$

$$\cos \theta = \frac{x}{r} < 0 \mid \sec < 0$$

$$\tan \theta < 0 \mid \cot \theta < 0$$

$$\rightarrow [table]$$

$$\theta \mid 0^\circ = 0 \mid 30^\circ = \frac{\pi}{6} \mid 45^\circ = \frac{\pi}{4} \mid 60^\circ = \frac{\pi}{3} \mid 90^\circ = \frac{\pi}{2}$$

$$\sin \theta \mid 0 \mid \frac{1}{2} \mid \frac{1}{\sqrt{2}} \mid \frac{\sqrt{3}}{2} \mid 1$$

$$\cos\theta \mid 1 \mid \frac{\sqrt{3}}{2} \mid \frac{1}{\sqrt{2}} \mid \frac{1}{2} \mid 0$$

$$\tan\theta \mid 0 \mid \frac{1}{\sqrt{3}} \mid 1 \mid \sqrt{3} \mid \infty$$

$$\cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

$$\csc 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

Result

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta \quad \checkmark$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta \quad \checkmark$$

$$\csc(-\theta) = -\csc\theta$$

[diagram: axes; $P(x, y)$ at angle θ above x -axis and $P'(x, -y)$ at angle $-\theta$ below, both at distance r from O]

$$\sin\theta = \frac{y}{r}, \sin(-\theta) = -\frac{y}{r}$$

$$= -\sin\theta$$

$$\cos\theta = \frac{x}{r}, \cos(-\theta) = \frac{x}{r}$$

$$= \cos\theta$$

Result

$$\sin(90^\circ - \theta) = \cos\theta$$

$$\cos(90^\circ - \theta) = \sin\theta$$

$$\tan(90^\circ - \theta) = \cot\theta$$

$$\cot(90^\circ - \theta) = \tan\theta$$

$$\sec(90^\circ - \theta) = \csc\theta$$

$$\csc(90^\circ - \theta) = \sec\theta.$$

[diagram: axes, right triangle $O-(x,0)-P(x, y)$, hypotenuse r , angle θ at O , angle $90^\circ - \theta$ at P , sides x and y]

$$\sin\theta = \frac{y}{r} \mid \sin(90^\circ - \theta) = \frac{x}{r}$$

$$= \cos\theta$$

$$\cos\theta = \frac{x}{r} \mid \cos(90^\circ - \theta) = \frac{y}{r}$$

$$= \sin\theta$$

Results:

$$\sin(90^\circ + \theta) = \cos\theta \quad \checkmark$$

$$\cos(90^\circ + \theta) = -\sin\theta$$

$$\tan(90^\circ + \theta) = -\cot\theta$$

$$\cot(90^\circ + \theta) = -\tan\theta$$

$$\sec(90^\circ + \theta) = -\csc\theta$$

$$\csc(90^\circ + \theta) = \sec\theta \quad \checkmark$$

$$\sin(180^\circ - \theta) = \sin\theta \quad \checkmark$$

$$\cos(180^\circ - \theta) = -\cos\theta$$

$$\tan(180^\circ - \theta) = -\tan\theta$$

[diagram: axes labelled 0 and 90°, II marked, arrow towards second quadrant]

[diagram: axes labelled 0 (360°), 90°, 180°, 270°; quadrants II and I marked]

$$\sin(180^\circ + \theta) = -\sin\theta$$

$$\sec(180^\circ + \theta) = -\sec\theta$$

$$\sin(270^\circ - \theta) = -\cos\theta$$

$$\tan(270^\circ - \theta) = \cot\theta$$

$$\csc(360^\circ - \theta) = -\csc\theta$$

$$\cot(360^\circ - \theta) = -\cot\theta$$

$$\tan(270^\circ + \theta) = -\cot\theta$$

$$\cos(360^\circ - \theta) = \cos\theta$$

$$\cos(270^\circ + \theta) = \sin\theta$$

[diagram: axes labelled $\frac{0}{360^\circ}$, 90°, 180°, 270°; quadrants I, II, III, IV marked]

Note The above results may be remembered as follows

(i) $A = 90^\circ \pm \theta$, $270^\circ \pm \theta$ then T.R must change

($\sin \leftrightarrow \cos$, $\tan \leftrightarrow \cot$, $\sec \leftrightarrow \csc$)

(ii) $A = 180^\circ \pm \theta$, $360^\circ \pm \theta$ then T.R does not change.

(iii) $\pm$ sign can be decided from the Quadrant in which A lies

Ex $\csc(90^\circ + \theta) = + \sec\theta$

$$\sec(180^\circ - \theta) = -\sec\theta$$

$$\cot(270^\circ + \theta) = -\tan\theta$$

$$\tan(360^\circ - \theta) = -\tan\theta$$

[diagram: axes labelled 0 (360°), 90°, 180°, 270°]

$$\tan(1440^\circ + 30^\circ) \text{ [1440}^\circ \text{ crossed out]}$$

$$= \frac{1}{\sqrt{3}}$$

$$1 \times 360^\circ = 360^\circ$$

$$2 \times 360^\circ = 720^\circ$$

$$1 \times 360^\circ = 360^\circ$$

$$2 \times 360^\circ = 720^\circ$$

$$4 \times 360^\circ = 1440^\circ$$

$$* \cos\left(\frac{33\pi}{4}\right) = \underline{\hspace{2cm}}$$

$$= \cos 1485^\circ$$

$$= \cos(1440^\circ + 45^\circ) \text{ [1440}^\circ \text{ crossed out, marked } 4 \times 360^\circ]$$

$$= \frac{1}{\sqrt{2}}$$

$$\text{(OR)} \cos\left(\frac{33\pi}{4}\right) = \cos\left(8\pi + \frac{\pi}{4}\right) \text{ [8}\pi \text{ crossed out]}$$

$$= \frac{1}{\sqrt{2}}$$

$$\frac{33\pi}{4} = 33 \times 45^\circ$$

$$=$$

$$45$$

$$33$$

$$135$$

$$135$$

$$1485$$

$$\rightarrow \sec\left(\frac{4\pi}{3}\right) = \underline{\hspace{2cm}}$$

$$= \sec(240^\circ)$$

$$= -2$$

$$\rightarrow \cot\left(\frac{11\pi}{3}\right) = \underline{\hspace{2cm}}$$

$$= \cot(660^\circ)$$

$$= \cot(360^\circ + 300^\circ) \text{ [360}^\circ \text{ crossed out]}$$

$$= -\cot 60^\circ$$

$$= -\frac{1}{\sqrt{3}}$$

$$\frac{\pi}{3} = 60^\circ$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\frac{\pi}{3} = 60^\circ$$

60

11

660

$$\tan 60^\circ = \sqrt{3}$$

→ **Note**

$$\sin 0^\circ = 0$$

$$\sin 90^\circ = 1$$

$$\sin 180^\circ = 0$$

$$\sin 270^\circ = -1$$

$$\sin 360^\circ = 0$$

$$\sin 450^\circ = 1$$

$$\sin 540^\circ = 0$$

$$\sin 630^\circ = -1$$

$$\sin 720^\circ = 0$$

$$\cos 0^\circ = 1$$

$$\cos 90^\circ = 0$$

$$\cos 180^\circ = -1$$

$$\cos 270^\circ = 0$$

$$\cos 360^\circ = 1$$

$$\cos 450^\circ = 0$$

$$\cos 540^\circ = -1$$

$$\cos 630^\circ = 0$$

$$\cos 720^\circ = 1$$

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

[diagram: axes with points (1, 0), (0, 1), (-1, 0), (0, -1) at distance 1 from origin (0, 0)]

Note $\sin \theta = 0 \Rightarrow \theta = n\pi, n \in \mathbb{Z}$

$$\cos \theta = 0 \Rightarrow \theta = \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}.$$

$$\tan \theta = 0 \Rightarrow \theta = n\pi$$

Ex $\sin \pi = \sin 2\pi = \sin 3\pi = \dots = 0$

$$\sin(-\pi) = \sin(-2\pi) = \dots = 0$$

Ex $\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = \cos\left(\frac{5\pi}{2}\right) = \cos\left(\frac{7\pi}{2}\right) = \dots = 0.$

Domain and ranges of Trigonometric functions

① $f(\theta) = \sin\theta$

[diagram: axes, point $P(x, y)$ at distance r from O at angle θ , with x and y marked]

$$\sin\theta = \frac{y}{r}$$

[diagram: y-axis with points $P(0, r)$ and $(0, -r)$, each at distance r from O]

Here $x^2 + y^2 = r^2$

$$-r \leq y \leq r$$

$$\Rightarrow -1 \leq \frac{y}{r} \leq 1$$

$$\Rightarrow -1 \leq \sin\theta \leq 1, \forall \theta \in \mathbb{R}$$

$$\therefore \text{Domain} = \mathbb{R}$$

$$\text{Range} = [-1, 1]$$

→ $f(\theta) = \tan\theta$

$$= \frac{\sin\theta}{\cos\theta}$$

where $\cos\theta \neq 0$

$$\Rightarrow \theta \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}.$$

$$\therefore \text{Domain} = \mathbb{R} - \left\{ \frac{(2n+1)\pi}{2}, n \in \mathbb{Z} \right\}$$

[diagram: axes, point $P(x, y)$ at distance r from O at angle θ]

$$\tan\theta = \frac{y}{x} \text{ where } x, y \in \mathbb{R}.$$

suppose $x = 1, y = 10, \tan\theta = 10$

$x = 1, y = 100, \tan\theta = 100$

$$\therefore \text{Range} = \mathbb{R} \Rightarrow \tan\theta \in \mathbb{R}$$

Note [table]

Function | Domain | Range

$\sin x$ | $\mathbb{R}$ | $[-1, 1]$

$\cos x$ | $\mathbb{R}$ | $[-1, 1]$

$\tan x = \frac{\sin x}{\cos x}$ | $\mathbb{R} - \left\{ \frac{(2n+1)\pi}{2}, n \in \mathbb{Z} \right\}$ | $\mathbb{R}$

$\cot x = \frac{\cos x}{\sin x}$ | $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$ | $\mathbb{R}.$

$$\sec x = \frac{1}{\cos x} \mid \mathbb{R} - \left\{ \frac{(2n+1)\pi}{2}, n \in \mathbb{Z} \right\} \mid (-\infty, -1] \cup (1, \infty)$$

$$\csc x = \frac{1}{\sin x} \mid \mathbb{R} - \{n\pi, n \in \mathbb{Z}\} \mid (-\infty, -1] \cup (1, \infty)$$

$$\cos x = 0$$

$$\Rightarrow x = \frac{(2n+1)\pi}{2},$$

$$n \in \mathbb{Z}$$

$$\sin x = 0$$

$$\Rightarrow x = n\pi,$$

$$n \in \mathbb{Z}$$

[sketch: axes with lines at 1 and -1 and curves lying outside the band between them]

Graph of $f(x) = \sin x$

[graph: sine wave through 0, peaks at 1 $(\frac{\pi}{2}, -\frac{3\pi}{2})$, troughs at -1 $(-\frac{\pi}{2}, \frac{3\pi}{2})$; x-axis marked $-2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$]

period of $\sin x = 2\pi$.

Graph of $\cos x$

[graph: cosine wave with maximum 1 at 0, minimum -1; x-axis marked $-2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$]

period = 2π .

Graph of $\tan x$

[graph: tangent curves through 0 and π , with vertical asymptotes at $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$]

period = π

Graph of $\cot x$

[graph: decreasing cot curves crossing the x-axis at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, with vertical asymptotes at 0, π , 2π]

period = π .

Graph of $\sec x$

[graph: U-shaped branch with minimum 1 at 0 between asymptotes $-\frac{\pi}{2}$ and $\frac{\pi}{2}$; inverted branch with maximum -1 at π between asymptotes $\frac{\pi}{2}$ and $\frac{3\pi}{2}$]

period = 2π

Graph of $\csc x$

[graph: U-shaped branch with minimum 1 at $\frac{\pi}{2}$ between asymptotes 0 and π ; inverted branch with maximum -1 at $\frac{3\pi}{2}$ between asymptotes π and 2π]

period = 2π .

Ex $x = \sin 1$, $y = \sin 1^\circ$ which one is greater

$1 = 1^\circ = 57^\circ$ (app)

$x = \sin 57^\circ$, $y = \sin 1^\circ$

[graph: sine curve with points 1° , 1 , 90° on x-axis; heights $\sin 1^\circ$ and $\sin 1$ marked on y-axis]

$\sin 1^\circ < \sin 1$

$\Rightarrow y < x$.

$\Rightarrow x > y$.

Ex $x = \tan 1$, $y = \tan 2$, $z = \tan 3$ then

which one is greater

Sol $1 = 1^\circ = 57^\circ$

$2 = 2^\circ = 114^\circ$

$3 = 3^\circ = 171^\circ$

[graph: tan curves with asymptotes at $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$; points 1, 2, 3 marked on x-axis with heights $\tan 1$ (positive), $\tan 3$ (small negative), $\tan 2$ (large negative)]

$\tan 1 > \tan 3 > \tan 2$

Result If A, B are complementary angles

i.e $A + B = 90^\circ$ then

(i) $\sin A = \cos B$

(ii) $\cos A = \sin B$

(iii) $\tan A = \cot B$

(iv) $\cot A = \tan B$

(v) $\sec A = \csc B$

(vi) $\csc A = \sec B$

LHS = $\sin A = \sin(90^\circ - B)$

= $\cos B$

(vii) $\sin^2 A + \sin^2 B = 1$

(viii) $\cos^2 A + \cos^2 B = 1$

(ix) $\tan A \cdot \tan B = 1$

(x) $\cot A \cdot \cot B = 1$

$$\begin{aligned}\text{LHS} &= \sin^2 A + \sin^2 B \\ &= \cos^2 B + \sin^2 B \\ &= 1 = \text{RHS}\end{aligned}$$

Ex $\sin 81^\circ = \cos 9^\circ$

$$\cos 70^\circ = \sin 20^\circ$$

$$\sin 85^\circ = \cos 5^\circ$$

$$\cos 89^\circ = \sin 1^\circ$$

$$\tan 80^\circ = \cot 10^\circ$$

$$\cot 80^\circ = \tan 10^\circ$$

Ex $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \dots + \sin^2 85^\circ = \underline{\hspace{1cm}}.$

$$5^\circ \quad 85^\circ$$

$$10^\circ \quad 80^\circ$$

$$15^\circ \quad 75^\circ$$

$$20^\circ \quad 70^\circ$$

$$25^\circ \quad 65^\circ$$

$$30^\circ \quad 60^\circ$$

$$35^\circ \quad 55^\circ$$

$$40^\circ \quad 50^\circ$$

$$45^\circ$$

$$\sin^2 5^\circ + \sin^2 85^\circ = 1$$

$$\text{G.E} = 8(1) + \sin^2 45^\circ$$

$$= 8 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= 8 + \frac{1}{2}$$

$$= 8\frac{1}{2}$$

$$= \frac{17}{2}$$

Result: If A, B are supplementary angles

i.e $A + B = 180^\circ$ then (i) $\cos A + \cos B = 0$

(ii) $\tan A + \tan B = 0$

Sol : (i) $\text{LHS} = \cos A + \cos B$

$$= \cos A + \cos(180^\circ - A)$$

$$= \cos A - \cos A$$

$$= 0$$

(ii) $\tan A + \tan B$

$$= \tan A + \tan(180^\circ - A)$$

$$= \tan A - \tan A$$

$$= 0$$

Note : If ABCD is a cyclic Quadrilateral

then $A + C = 180^\circ$, $B + D = 180^\circ$

[diagram: quadrilateral ABCD inscribed in a circle]

Ex If ABCD is a cyclic Quadrilateral

then $\cos A + \cos B + \cos C + \cos D = \underline{\hspace{2cm}}$

$$= \cos A + \cos B + \cos(180^\circ - A) + \cos(180^\circ - B)$$

$$= \cos A + \cos B - \cos A - \cos B = 0$$

Results : 1) $\sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta = 1 - \sin^2 \theta \cos^2 \theta$

2) $\sin^4 \theta + \cos^4 \theta = 1 - 2\sin^2 \theta \cos^2 \theta$

3) $\sin^6 \theta + \cos^6 \theta = 1 - 3\sin^2 \theta \cos^2 \theta$.

Proof (i) $\sin^2 \theta + \cos^4 \theta = \sin^2 \theta + \cos^2 \theta \cdot \cos^2 \theta$

$$= \sin^2 \theta + \cos^2 \theta (1 - \sin^2 \theta)$$

$$= \sin^2 \theta + \cos^2 \theta - \sin^2 \theta \cos^2 \theta$$

$$= 1 - \sin^2 \theta \cos^2 \theta$$

2) $\sin^4 \theta + \cos^4 \theta = (\sin^2 \theta)^2 + (\cos^2 \theta)^2$

$$= (\sin^2 \theta + \cos^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta$$

$$= 1 - 2\sin^2 \theta \cos^2 \theta$$

$$a^2 + b^2 = (a + b)^2 - 2ab$$

$$a^3 + b^3 = (a + b)^3$$

$$- 3ab(a + b)$$

Note If the given no's are +ve then their $A.M \geq G.M \geq H.M$

Result

$$\text{If } x > 0 \text{ then } x + \frac{1}{x} \geq 2$$

$$\text{If } x < 0 \text{ then } x + \frac{1}{x} \leq -2$$

Proof Given $x > 0$

Consider $x, \frac{1}{x}$

$$A.M \geq G.M \Rightarrow \frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \left(\frac{1}{x}\right)}$$

$$\Rightarrow x + \frac{1}{x} \geq 2$$

For $x < 0$

Consider $-x, -\frac{1}{x}$

These are +ve

$$\Rightarrow \text{A.M} \geq \text{G.M}$$

$$\Rightarrow \frac{-x - \frac{1}{x}}{2} \geq \sqrt{\left(-x\right)\left(-\frac{1}{x}\right)}$$

$$\Rightarrow -x - \frac{1}{x} \geq 2$$

$$\Rightarrow x + \frac{1}{x} \leq -2$$

Result 1) $\sin^2 x + \csc^2 x \geq 2$

2) $\cos^2 x + \sec^2 x \geq 2$

3) $\tan^2 x + \cot^2 x \geq 2$

Proof Consider $\sin^2 x, \csc^2 x$

A.M $\geq$ G.M

$$\Rightarrow \frac{\sin^2 x + \csc^2 x}{2} \geq \sqrt{\sin^2 x \cdot \csc^2 x}$$

$$\Rightarrow \sin^2 x + \csc^2 x \geq 2$$

Note $x \in \mathbb{R}, \sqrt{x^2} = x$ if $x \geq 0$

$= -x$ if $x < 0$.

Ex $\sqrt{1 - \sin^2 100^\circ} = \underline{\hspace{2cm}}$

$$= \sqrt{\cos^2 100^\circ}$$

$$= -\cos 100^\circ$$

$$\because \cos 100^\circ < 0$$

$$100^\circ \in Q_2$$

Ratio and proportion

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{a}{b} = \frac{c}{d} = \frac{a+cm}{b+dm}$$

■ Componendo and dividendo

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\text{(or)} \frac{a-b}{a+b} = \frac{c-d}{c+d}$$

Compound angles

Formulae

$$\checkmark \text{ I } \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\checkmark \text{ II } \sin(A + B) + \sin(A - B) = 2 \sin A \cos B$$

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$$

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$

$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$$

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$$

$$A + B = C$$

$$A - B = D$$

$$2A = C + D$$

$$A = \frac{C + D}{2}$$

$$B = \frac{C - D}{2}$$

$$\text{III } \sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$\sin C - \sin D = 2 \cos\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

$$\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

$$\text{IV } \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

$$\begin{aligned} &= \frac{\frac{1}{\cot A} + \frac{1}{\cot B}}{1 - \frac{1}{\cot A \cot B}} \\ &= \frac{\cot B + \cot A}{\cot A \cot B - 1} \end{aligned}$$

$$\begin{aligned} \text{V } \sin(A + B) \cdot \sin(A - B) &= \sin^2 A - \sin^2 B \quad \checkmark \\ &= \cos^2 B - \cos^2 A. \end{aligned}$$

$$\begin{aligned} \cos(A + B) \cdot \cos(A - B) &= \cos^2 A - \sin^2 B \quad \checkmark \\ &= \cos^2 B - \sin^2 A. \end{aligned}$$

$$\tan(A + B) \cdot \tan(A - B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \cdot \tan^2 B}.$$

$$\begin{aligned} \text{VI } \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} &= \frac{1 + \tan \theta}{1 - \tan \theta} = \tan\left(\frac{\pi}{4} + \theta\right) = \cot\left(\frac{\pi}{4} - \theta\right) \\ \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} &= \frac{1 - \tan \theta}{1 + \tan \theta} = \tan\left(\frac{\pi}{4} - \theta\right) = \cot\left(\frac{\pi}{4} + \theta\right) \end{aligned}$$

$$\begin{aligned} \text{VII } \sin(A + B + C) &= \Sigma \sin A \cos B \cos C - \Pi \sin A \\ &= \sin[(A + B) + C] \\ &= \sin(A + B) \cos C + \cos(A + B) \sin C \\ &= (\sin A \cos B + \cos A \sin B) \cos C + (\cos A \cos B - \sin A \sin B) \sin C \\ &= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C \\ &\quad - \sin A \sin B \sin C \\ &= \Sigma \sin A \cos B \cos C - \Pi \sin A. \end{aligned}$$

$$\cos(A + B + C) = \Pi \cos A - \Sigma \cos A \sin B \sin C.$$

$$\begin{aligned} \tan(A + B + C) &= \frac{\Sigma \tan A - \Pi \tan A}{1 - \Sigma \tan A \tan B} \\ &= \frac{(\tan A + \tan B + \tan C) - (\tan A \tan B \tan C)}{1 - [\tan A \tan B + \tan B \tan C + \tan C \tan A]} \end{aligned}$$

$$\text{VIII } \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\tan 15^\circ = 2 - \sqrt{3}$$

$$\cot 15^\circ = 2 + \sqrt{3}$$

$$\text{Hint } \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$A = 60^\circ, B = 45^\circ$$

Results : In a $\triangle ABC$ i.e $A + B + C = 180^\circ$ then

$$\checkmark \text{ (i) } \Sigma \tan A = \Pi \tan A$$

$$\checkmark \text{ (ii) } \Sigma \cot A \cot B = 1$$

$$\checkmark \text{ (iii) } \Sigma \tan A \tan B = \frac{1 + \cos A \cos B \cos C}{\cos A \cos B \cos C}$$

Proof (i) $A + B + C = 180^\circ$

$$\Rightarrow A + B = 180^\circ - C$$

$$\Rightarrow \tan(A + B) = \tan(180^\circ - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \tan A + \tan B$$

$$= -\tan C + \tan A \tan B \tan C$$

$$\Rightarrow \Sigma \tan A$$

$$= \Pi \tan A$$

$$(ii) A + B + C = 180^\circ$$

$$\Rightarrow A + B = 180^\circ - C$$

$$\Rightarrow \cot(A + B) = \cot(180^\circ - C)$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot B + \cot A} = -\cot C$$

$$\Rightarrow \cot A \cot B - 1 = -\cot B \cot C - \cot A \cot C$$

$$\Rightarrow \Sigma \cot A \cot B = 1$$

$$(iii) \cos(A + B + C) = \Pi \cos A - \Sigma \cos A \sin B \sin C$$

$$\Rightarrow \frac{\cos(A + B + C)}{\cos A \cos B \cos C} = 1 - \Sigma \tan B \tan C.$$

$$\Rightarrow -\frac{1}{\cos A \cos B \cos C} = 1 - \Sigma \tan B \tan C$$

$$\Rightarrow \Sigma \tan B \tan C = 1 + \frac{1}{\cos A \cos B \cos C}$$

$$\Rightarrow \Sigma \tan A \tan B = \frac{\cos A \cos B \cos C + 1}{\cos A \cos B \cos C}$$

$$A + B + C = 180^\circ$$

$$\cos(A + B + C)$$

$$= \cos 180^\circ$$

$$= -1$$

Result : If $A + B + C = 90^\circ$ then

$$(i) \Sigma \cot A = \Pi \cot A$$

$$(ii) \Sigma \tan A \tan B = 1$$

$$\text{Hint } A + B = 90^\circ - C$$

$$\Rightarrow \cot(A + B) = \cot(90^\circ - C)$$

Result If $A + B = 45^\circ$ (or) 225° then

$$\checkmark (i) (1 + \tan A)(1 + \tan B) = 2$$

$$\checkmark (ii) (1 - \cot A)(1 - \cot B) = 2$$

$$\checkmark (iii) (1 + \cot A)(1 + \cot B) = 2 \cot A \cot B$$

Replace $\tan A, \tan B$ by $\cot A, \cot B$

Proof

$$A + B = 45^\circ$$

$$\Rightarrow \tan(A + B) = \tan 45^\circ$$

$$\begin{aligned} \Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} &= 1 \\ \Rightarrow \tan A + \tan B &= 1 - \tan A \tan B \\ \Rightarrow \tan A + \tan B + \tan A \tan B &= 1 \\ \Rightarrow 1 + \tan A + \tan B + \tan A \tan B &= 2 \\ \Rightarrow (1 + \tan A) + \tan B(1 + \tan A) &= 2 \\ \Rightarrow (1 + \tan A)(1 + \tan B) &= 2 \end{aligned}$$

Ex $(1 + \tan 22^\circ)(1 + \tan 23^\circ) = 2$

Ex $\sin^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ = \underline{\hspace{2cm}}$

$$\begin{aligned} &= \sin(52\frac{1}{2}^\circ + 22\frac{1}{2}^\circ) \sin(52\frac{1}{2}^\circ - 22\frac{1}{2}^\circ) \\ &= \sin 75^\circ \sin 30^\circ \\ &= \cos 15^\circ \sin 30^\circ \\ &= \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) \left(\frac{1}{2}\right) = \frac{\sqrt{3} + 1}{4\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \sin^2 A - \sin^2 B &= \sin(A + B) \sin(A - B) \end{aligned}$$

Ex P.T. $\frac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \cot 36^\circ$

$$\begin{aligned} &= \tan(45^\circ + 9^\circ) \\ &= \tan 54^\circ \\ &= \cot 36^\circ \end{aligned}$$

$$\begin{aligned} \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} &= \frac{1 + \tan \theta}{1 - \tan \theta} \\ &= \tan\left(\frac{\pi}{4} + \theta\right) \end{aligned}$$

Ex $\tan 100^\circ + \tan 125^\circ + \tan 100^\circ \tan 125^\circ = \underline{\hspace{2cm}}.$

Sol $100^\circ + 125^\circ = 225^\circ$

$$\begin{aligned} \Rightarrow \tan(100^\circ + 125^\circ) &= \tan 225^\circ \\ \Rightarrow \frac{\tan 100^\circ + \tan 125^\circ}{1 - \tan 100^\circ \tan 125^\circ} &= 1 \\ \Rightarrow \tan 100^\circ + \tan 125^\circ &= 1 - \tan 100^\circ \tan 125^\circ \\ \Rightarrow \tan 100^\circ + \tan 125^\circ + \tan 100^\circ \tan 125^\circ &= 1 \end{aligned}$$

Ex: prove that $\tan 70^\circ - \tan 20^\circ = 2 \tan 50^\circ$

Sol $70^\circ - 20^\circ = 50^\circ$

$$\begin{aligned} \Rightarrow \tan(70^\circ - 20^\circ) &= \tan 50^\circ \\ \Rightarrow \frac{\tan 70^\circ - \tan 20^\circ}{1 + \tan 70^\circ \tan 20^\circ} &= \tan 50^\circ \end{aligned}$$

$$\Rightarrow \frac{\tan 70^\circ - \tan 20^\circ}{2} = \tan 50^\circ$$

$$\Rightarrow \tan 70^\circ - \tan 20^\circ = 2 \tan 50^\circ$$

$$A + B = 90^\circ$$

$$\tan A \cdot \tan B = 1$$

Multiple and Sub multiple angles

$$\checkmark 1) \sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}.$$

$$\cos 2A = \cos^2 A - \sin^2 A.$$

$$= 1 - 2 \sin^2 A.$$

$$= 2 \cos^2 A - 1$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$$\sin 4A = 2 \sin 2A \cos 2A$$

$$\sin 8A = 2 \sin 4A \cos 4A$$

$$\sin 16A = 2 \sin 8A \cos 8A$$

$$\sin 20^\circ = 2 \sin 10^\circ \cos 10^\circ$$

$$\sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)$$

$$\sin 100^\circ = 2 \sin 50^\circ \cos 50^\circ$$

$$2 \sin 2y \cos 2y = \sin 4y$$

$$2 \sin 8x \cos 8x = \sin 16x$$

$$2) 1 - \cos 2A = 2 \sin^2 A.$$

$$1 + \cos 2A = 2 \cos^2 A$$

$$\frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A.$$

$$\Rightarrow \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

$$\sin A = \pm \sqrt{\frac{1 - \cos 2A}{2}}$$

$$\cos A = \pm \sqrt{\frac{1 + \cos 2A}{2}}$$

$$\tan A = \pm \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}.$$

$$3) \sin 3A = 3\sin A - 4\sin^3 A.$$

$$\cos 3A = 4\cos^3 A - 3\cos A.$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

$$4) \sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = \frac{2\tan\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$$

$$\begin{aligned}\cos A &= \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right) \\ &= 1 - 2\sin^2\left(\frac{A}{2}\right)\end{aligned}$$

$$= 2\cos^2\left(\frac{A}{2}\right) - 1 = \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$$

$$\tan A = \frac{2\tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$$

$$5) 1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$$

$$1 + \cos A = 2\cos^2\left(\frac{A}{2}\right)$$

$$\Rightarrow \sin^2\left(\frac{A}{2}\right) = \frac{1 - \cos A}{2}$$

$$\cos^2\left(\frac{A}{2}\right) = \frac{1 + \cos A}{2}$$

$$\tan^2\left(\frac{A}{2}\right) = \frac{1 - \cos A}{1 + \cos A}$$

$$\sin\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 + \cos A}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}.$$

$$6) \sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$$

$$\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$$

$$\tan A = \frac{3\tan^2\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\textcircled{7} \sin 18^\circ = \frac{\sqrt{5} - 1}{4} = \cos 72^\circ$$

$$\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \sin 72^\circ$$

$$\sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \cos 54^\circ$$

$$\cos 36^\circ = \frac{\sqrt{5} + 1}{4} = \sin 54^\circ.$$

$$\textcircled{8} \sin 22\frac{1}{2}^\circ = \frac{\sqrt{2} - 1}{2\sqrt{2}} = \cos 67\frac{1}{2}^\circ$$

$$\cos 22\frac{1}{2}^\circ = \frac{\sqrt{2} + 1}{2\sqrt{2}} = \sin 67\frac{1}{2}^\circ$$

$$\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1 = \cot 67\frac{1}{2}^\circ$$

$$\cot 22\frac{1}{2}^\circ = \sqrt{2} + 1 = \tan 67\frac{1}{2}^\circ.$$

$$\text{Hint } \sin\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$A = 45^\circ \sin 22\frac{1}{2}^\circ = \sqrt{\frac{1 - \cos 45^\circ}{2}}$$

Results :

$$\textcircled{1} \tan 7\frac{1}{2}^\circ = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1) = \sqrt{6} - \sqrt{4} - \sqrt{3} + \sqrt{2}$$

$$\cot 7\frac{1}{2}^\circ = (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1)$$

$$\tan 37\frac{1}{2}^\circ = (\sqrt{3} - \sqrt{2})(\sqrt{2} + 1)$$

$$\cot 37\frac{1}{2}^\circ = (\sqrt{3} + \sqrt{2})(\sqrt{2} - 1)$$

$$\begin{aligned} \text{Proof } \tan 7\frac{1}{2}^\circ &= \sin 7\frac{1}{2}^\circ \frac{1}{\cos 7\frac{1}{2}^\circ} = 2\sin^2 7\frac{1}{2}^\circ \frac{1}{2\sin 7\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ} \\ &= \frac{2\sin^2 7\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ}{2\cos^2 7\frac{1}{2}^\circ} = \frac{\sin 15^\circ}{1 + \cos 15^\circ} \text{ [this line crossed out]} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}} \text{ [crossed out]} \\
 &= \frac{\sqrt{3}-1}{2\sqrt{2} + \sqrt{3}+1} \text{ [crossed out]} \\
 &= \\
 &= \sqrt{6} + \sqrt{2} - \sqrt{3} - 2 \\
 &= (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1) \\
 &= \frac{1 - \cos 15^\circ}{\sin 15^\circ} \\
 &= \frac{1 - \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} \\
 &= \left(\frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1}\right) \times \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right) \\
 &= \frac{2\sqrt{6} + 2\sqrt{2} - 3 - \sqrt{3} - \sqrt{3} - 1}{2} \\
 &= \frac{2\sqrt{6} + 2\sqrt{2} - 2\sqrt{3} - 4}{2}
 \end{aligned}$$

Result ② (i) $\cot A - \tan A = 2\cot 2A$

(ii) $\cot A + \tan A = 2\csc 2A$.

✓ (iii) $\cot A - \tan A - 2\tan 2A - 4\tan 4A = 8\cot 8A$.

Proof (i) LHS = $\cot A - \tan A$

$$\begin{aligned}
 &= \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} \\
 &= \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} \\
 &= \frac{\cos 2A}{\left(\frac{1}{2}\right) \cdot (\sin 2A)} = 2\cot 2A.
 \end{aligned}$$

$$a - x = b$$

$$a - b = x$$

(iii) $\cot A - \tan A = 2\cot 2A$

$$\Rightarrow \tan A = \cot A - 2\cot 2A. \quad \checkmark$$

$$\text{LHS} = \cot A - \tan A - 2\tan 2A - 4\tan 4A.$$

$$= \cot A - (\cot A - 2\cot 2A)$$

$$- 2(\cot 2A - 2\cot 4A) - 4(\cot 4A - 2\cot 8A)$$

$$= \cot A - \cot A + 2\cot 2A - 2\cot 2A$$

$$+ 4\cot 4A - 4\cot 4A + 8\cot 8A$$

$$= 8\cot 8A.$$

Result ③ (i) $\tan\left(\frac{\theta}{2}\right)(1 + \sec\theta) = \tan\theta$

(ii) $\tan\left(\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec2\theta) = \tan2\theta$

(iii) $\tan\left(\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec2\theta)(1 + \sec4\theta) = \tan4\theta$

Proof (i) LHS = $\tan\left(\frac{\theta}{2}\right)(1 + \sec\theta)$

$$\begin{aligned} &= \left(\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)\left(1 + \frac{1}{\cos\theta}\right) \\ &= \left(\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)\left(\frac{1 + \cos\theta}{\cos\theta}\right) \\ &= \left(\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right) \cdot \frac{2\cos^2\left(\frac{\theta}{2}\right)}{\cos\theta} \\ &= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \end{aligned}$$

Result ④ (i) $\cos\theta \cdot \cos2\theta \cos4\theta = \frac{\sin8\theta}{8\sin\theta}$

(ii) $\cos\theta \cos2\theta \cos4\theta \cos8\theta = \frac{\sin16\theta}{16\sin\theta}$

(iii) $\cos\theta \cos2\theta \cos2^2\theta \cdot \cos2^3\theta \cdots \cos2^n\theta = \frac{\sin2^{n+1}\theta}{2^{n+1} \cdot \sin\theta}$

Proof (i) LHS = $\cos\theta \cos2\theta \cos4\theta$

$$\begin{aligned} &= \left(\frac{1}{8\sin\theta}\right)(8\sin\theta \cdot \cos\theta \cos2\theta \cos4\theta) \\ &= \left(\frac{1}{8\sin\theta}\right)[4(2\sin\theta\cos\theta)\cos2\theta \cos4\theta] \\ &= \left(\frac{1}{8\sin\theta}\right)[4\sin2\theta \cos2\theta \cos4\theta] \\ &= \left(\frac{1}{8\sin\theta}\right)[2(2\sin2\theta\cos2\theta)\cos4\theta] \\ &= \left(\frac{1}{8\sin\theta}\right)(2\sin4\theta\cos4\theta) \\ &= \left(\frac{1}{8\sin\theta}\right)\sin8\theta \\ &= \frac{\sin8\theta}{8\sin\theta} \end{aligned}$$

Ex : prove that $\cos(\frac{2\pi}{7})\cos(\frac{4\pi}{7})\cos(\frac{6\pi}{7}) = \frac{1}{8}$

Sol LHS = $\cos(\frac{2\pi}{7})\cos(\frac{4\pi}{7})\cos(\frac{6\pi}{7})$
 $= -\cos(\frac{\pi}{7})\cos(\frac{2\pi}{7})\cos(\frac{4\pi}{7})$
 $= -[\frac{\sin(\frac{8\pi}{7})}{8\sin(\frac{\pi}{7})}]$
 $= -\frac{\sin(\pi + \frac{\pi}{7})}{8\sin(\frac{\pi}{7})} = -\frac{-\sin(\frac{\pi}{7})}{8\sin(\frac{\pi}{7})} = \frac{1}{8}$
 $\therefore \cos(\frac{6\pi}{7})$
 $= \cos(\pi - \frac{\pi}{7})$
 $= -\cos(\frac{\pi}{7})$

Result ⑤ If $a_1, a_2, \dots, a_n$ are in A.P with common difference 'd' then

(i) $\cos a_1 + \cos a_2 + \cos a_3 + \dots + \cos a_n = \cos(\frac{a_1 + a_n}{2}) \frac{\sin(\frac{nd}{2})}{\sin(\frac{d}{2})}$

(ii) $\sin a_1 + \sin a_2 + \sin a_3 + \dots + \sin a_n = \sin(\frac{a_1 + a_n}{2}) \frac{\sin(\frac{nd}{2})}{\sin(\frac{d}{2})}$

Hint

multiply and divide with $2\sin(\frac{d}{2})$

Ex $\cos(\frac{2\pi}{7}) + \cos(\frac{4\pi}{7}) + \cos(\frac{6\pi}{7}) = \underline{\hspace{2cm}}$

$\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7} \dots$ are in A.P, $d = \frac{2\pi}{7}$

$\frac{d}{2} = \frac{\pi}{7}$

Hint multiply and divide with

$2\sin(\frac{d}{2}) = 2\sin(\frac{\pi}{7})$

G.E = $(\frac{1}{2\sin(\frac{\pi}{7})})[2\cos(\frac{2\pi}{7})\sin(\frac{\pi}{7}) + 2\cos(\frac{4\pi}{7})\sin(\frac{\pi}{7}) + 2\cos(\frac{6\pi}{7})\sin(\frac{\pi}{7})]$

$= (\frac{1}{2\sin(\frac{\pi}{7})})[(\sin(\frac{3\pi}{7}) - \sin(\frac{\pi}{7})) + (\sin(\frac{5\pi}{7}) - \sin(\frac{3\pi}{7}))$

$+ (\sin\pi - \sin(\frac{5\pi}{7}))]$

$$= -\frac{\sin\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{\pi}{7}\right)} = -\frac{1}{2}$$

$$2\cos A \sin B$$

$$= \sin(A + B) - \sin(A - B)$$

Note : Let $a = \cos\left(\frac{2\pi}{7}\right)$, $b = \cos\left(\frac{4\pi}{7}\right)$, $c = \cos\left(\frac{6\pi}{7}\right)$ then

✓ (i) $a + b + c = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$

(ii) $ab + bc + ca = \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)$
 $= -\frac{1}{2}$

✓ (iii) $abc = \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right) = \frac{1}{8}$

Proof LHS = $\left(\frac{1}{2}\right)[2\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{6\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right) + 2\cos\left(\frac{6\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)]$
 $= \left(\frac{1}{2}\right)\left(\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{10\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right)\right)$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$= \left(\frac{1}{2}\right)[2(\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right))]$$

$$= \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$$

$$= -\frac{1}{2}$$

$$\cos\left(\frac{8\pi}{7}\right)$$

$$= \cos\left(2\pi - \frac{6\pi}{7}\right)$$

$$= \cos\left(\frac{6\pi}{7}\right)$$

$$\cos\left(\frac{10\pi}{7}\right)$$

$$= \cos\left(2\pi - \frac{4\pi}{7}\right)$$

$$= \cos\left(\frac{4\pi}{7}\right)$$

Ex $\sin\left(\frac{\pi}{7}\right) \sin\left(\frac{2\pi}{7}\right) \sin\left(\frac{3\pi}{7}\right) = \frac{\sqrt{7}}{8}$

Sol let $k = \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{2\pi}{7}\right) \sin\left(\frac{3\pi}{7}\right)$

$$k^2 = \sin^2\left(\frac{\pi}{7}\right) \sin^2\left(\frac{2\pi}{7}\right) \sin^2\left(\frac{3\pi}{7}\right)$$

$$= \left(\frac{1 - \cos\left(\frac{2\pi}{7}\right)}{2}\right) \left(\frac{1 - \cos\left(\frac{4\pi}{7}\right)}{2}\right) \left(\frac{1 - \cos\left(\frac{6\pi}{7}\right)}{2}\right)$$

$$\begin{aligned}
 &= \left(\frac{1}{8}\right)(1-a)(1-b)(1-c) \\
 &= \left(\frac{1}{8}\right)[1^3 - (a+b+c)(1)^2 + (ab+bc+ca)1 - abc] \\
 &= \left(\frac{1}{8}\right)\left[1 - \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) - \frac{1}{8}\right] = \left(\frac{1}{8}\right)\left(\frac{7}{8}\right) = \frac{7}{64}
 \end{aligned}$$

$$\begin{aligned}
 (x-a)(x-b)(x-c) \\
 = x^3 - (a+b+c)x^2
 \end{aligned}$$

$$+ (ab+bc+ca)x$$

$$- abc.$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}.$$

$$\text{where } a = \cos\left(\frac{2\pi}{7}\right)$$

$$b = \cos\left(\frac{4\pi}{7}\right)$$

$$c = \cos\left(\frac{6\pi}{7}\right)$$

Result If $x = 60^\circ$ (or) 120° (or) 240° (or) 300° then

$$(i) \sin\theta \sin(x+\theta) \sin(x-\theta) = \left(\frac{1}{4}\right)\sin 3\theta$$

$$(ii) \cos\theta \cos(x+\theta) \cos(x-\theta) = \left(\frac{1}{4}\right)\cos 3\theta$$

$$(iii) \tan\theta \tan(x+\theta) \cdot \tan(x-\theta) = \tan 3\theta$$

$$(iv) \cot\theta \cot(x+\theta) \cot(x-\theta) = \cot 3\theta$$

$$\text{Ex: } \sin\theta \sin(60^\circ + \theta) \sin(60^\circ - \theta) = \left(\frac{1}{4}\right)\sin 3\theta$$

$$= \sin\theta [\sin^2 60^\circ - \sin^2 \theta]$$

$$= \sin\theta \left[\frac{3}{4} - \sin^2 \theta\right]$$

$$\sin(A+B) \cdot \sin(A-B)$$

$$= \sin^2 A - \sin^2 B$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$= \sin\theta \left[\frac{3 - 4\sin^2 \theta}{4}\right]$$

$$= \left(\frac{1}{4}\right)(3\sin\theta - 4\sin^3 \theta)$$

$$= \left(\frac{1}{4}\right)\sin 3\theta.$$

$$= \text{RHS}$$

Ex $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \underline{\hspace{1cm}}$.

$$= \left(\frac{1}{2}\right) [\cos 20^\circ \cos(60^\circ + 20^\circ) \cos(60^\circ - 20^\circ)]$$

$$= \left(\frac{1}{2}\right) \cdot \left(\frac{1}{4}\right) \cos[3(20^\circ)]$$

$$= \left(\frac{1}{8}\right) \cos 60^\circ = \left(\frac{1}{8}\right) \times \left(\frac{1}{2}\right) = \frac{1}{16}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\cos \theta \cos(60^\circ + \theta)$$

$$\cos(60^\circ - \theta)$$

$$= \left(\frac{1}{4}\right) \cos 3\theta$$

Ex If $A = \tan 6^\circ \tan 42^\circ$, $B = \cot 66^\circ \cot 78^\circ$ then

then $\frac{A}{B} = \underline{\hspace{1cm}}$

Sol $\frac{A}{B} = \frac{\tan 6^\circ \tan 42^\circ}{\cot 66^\circ \cot 78^\circ}$

$$= \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$$

$$= \frac{\tan 6^\circ \tan(60^\circ + 6^\circ) \tan(60^\circ - 6^\circ)}{\tan(60^\circ - 6^\circ)} \cdot \frac{\tan 18^\circ \tan(60^\circ - 18^\circ) \tan(60^\circ + 18^\circ)}{\tan 18^\circ}$$

$$= \frac{\tan 18^\circ \cdot \tan 54^\circ}{\tan 54^\circ \tan 18^\circ}$$

$$\frac{A}{B} = 1 \Rightarrow A = B$$

Result If $x = 60^\circ$ (or) 120° (or) 240° (or) 300° then

✓ (i) $\tan \theta + \tan(x + \theta) - \tan(x - \theta) = 3 \tan 3\theta$

$$= 3 \tan \theta \tan(x + \theta) \tan(x - \theta)$$

(ii) $\cot \theta + \cot(x + \theta) - \cot(x - \theta) = 3 \cot 3\theta$

$$= 3 \cot \theta \cot(x + \theta) \cot(x - \theta)$$

(iii) $\cot(x + \theta) \cot(x - \theta) + \cot(x - \theta) \cot \theta - \cot \theta \cot(x + \theta) = 3$

(iv) $\tan(x + \theta) \tan(x - \theta) + \tan(x - \theta) \tan \theta - \tan \theta \tan(x + \theta) = 3$

Proof (i) LHS = $\tan \theta + \tan(60^\circ + \theta) - \tan(60^\circ - \theta)$

$$= \tan \theta + \left(\frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \cdot \tan \theta}\right) - \left(\frac{\sqrt{3} - \tan \theta}{1 + \sqrt{3} \tan \theta}\right)$$

$$= 3 \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}\right)$$

$$= 3 \tan 3\theta$$

$\tan(A + B)$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\checkmark \rightarrow \tan\theta + \tan(60^\circ + \theta) - \tan(60^\circ - \theta) = 3\tan 3\theta = 3\tan\theta \tan(60^\circ + \theta) \tan(60^\circ - \theta)$$

$$\checkmark \rightarrow \cot(60^\circ + \theta) \cot(60^\circ - \theta) + \cot(60^\circ - \theta) \cot\theta - \cot\theta \cot(60^\circ + \theta) = 3$$

$$\rightarrow \cot 76^\circ \cot 44^\circ + \cot 44^\circ \cot 16^\circ - \cot 16^\circ \cot 76^\circ = 3$$

$$\text{Result 1) } \cos\theta + \cos(120^\circ + \theta) + \cos(120^\circ - \theta) = 0$$

$$2) \cos^3\theta + \cos^3(120^\circ + \theta) + \cos^3(120^\circ - \theta) = \left(\frac{3}{4}\right)\cos 3\theta$$

$$\text{Sol 1) LHS} = \cos\theta + \cos(120^\circ + \theta) + \cos(120^\circ - \theta)$$

$$= \cos\theta + 2\cos 120^\circ \cos\theta$$

$$= \cos\theta + 2\left(-\frac{1}{2}\right)\cos\theta$$

$$= \cos\theta - \cos\theta$$

$$= 0 = \text{RHS}$$

$$2) \text{ If } a + b + c = 0$$

$$\Rightarrow a^3 + b^3 + c^3 = 3abc$$

$$\cos^3\theta + \cos^3(120^\circ + \theta) + \cos^3(120^\circ - \theta) = 3\cos\theta \cos(120^\circ + \theta) \cos(120^\circ - \theta) = \left(\frac{3}{4}\right)\cos 3\theta$$

$$\cos(A + B) + \cos(A - B)$$

$$= 2\cos A \cos B$$

$$\text{Result 1) } \sin^2\theta + \sin^2(60^\circ + \theta) + \sin^2(60^\circ - \theta) = \frac{3}{2}$$

$$2) \cos^2\theta + \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) = \frac{3}{2}$$

$$\text{Proof 1) LHS} = \sin^2\theta + \sin^2(60^\circ + \theta) + \sin^2(60^\circ - \theta)$$

$$= \frac{1 - \cos 2\theta}{2} + \frac{1 - \cos(120^\circ + 2\theta)}{2} + \frac{1 - \cos(120^\circ - 2\theta)}{2}$$

$$= \frac{3}{2} - \left(\frac{1}{2}\right)[\cos 2\theta + \cos(120^\circ + 2\theta) + \cos(120^\circ - 2\theta)]$$

$$= \frac{3}{2} - \left(\frac{1}{2}\right)(0)$$

$$= \frac{3}{2} = \text{RHS}$$

$$\text{Ex } \sin^2 10^\circ + \sin^2 70^\circ + \sin^2 50^\circ = \frac{3}{2}$$

Result If $A + B = 60^\circ$ then

$$(i) \cos^2 A + \cos^2 B - \cos A \cos B = \frac{3}{4}$$

$$(ii) \sin^2 A + \sin^2 B + \sin A \sin B = \frac{3}{4}$$

Proof Given $A + B = 60^\circ$

$$\begin{aligned} \cos^2 A + \cos^2 B - \cos A \cos B &= \cos^2 A + 1 - \sin^2 B - \left(\frac{1}{2}\right)(2\cos A \cos B) \\ &= 1 + \cos(A + B) \cos(A - B) - \left(\frac{1}{2}\right)[\cos(A + B) + \cos(A - B)] \\ &= 1 + \cos 60^\circ \cos(A - B) - \left(\frac{1}{2}\right)[\cos 60^\circ + \cos(A - B)] \\ &= 1 + \left(\frac{1}{2}\right)\cos(A - B) - \frac{1}{4} - \left(\frac{1}{2}\right)\cos(A - B) = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

Ex : $\cos^2 76^\circ + \cos^2 16^\circ - \cos 76^\circ \cos 16^\circ = \frac{3}{4}$

If $A - B = 60^\circ$

then $\cos^2 A + \cos^2 B - \cos A \cos B = \frac{3}{4}$

Note

$$\sqrt{1 + \sin 2A} = |\cos A + \sin A|$$

$$\sqrt{1 - \sin 2A} = |\cos A - \sin A|$$

$$\sqrt{1 + \sin A} = \left| \cos\left(\frac{A}{2}\right) + \sin\left(\frac{A}{2}\right) \right|$$

$$\sqrt{1 - \sin A} = \left| \cos\left(\frac{A}{2}\right) - \sin\left(\frac{A}{2}\right) \right|$$

Hint $\sqrt{1 + \sin 2A}$

$$= \sqrt{\cos^2 A + \sin^2 A + 2\sin A \cos A}$$

$$= \sqrt{(\cos A + \sin A)^2}$$

$$= |\cos A + \sin A|$$

$$\sqrt{x^2} = |x|$$

Note If $C = \cos(\frac{A}{2})$, $S = \sin(\frac{A}{2})$ then

[diagram: two crossing lines through a point, ends labelled $\frac{\pi}{4} (\frac{9\pi}{4}), \frac{3\pi}{4}, \frac{5\pi}{4}, -\frac{\pi}{4} (\frac{7\pi}{4})$, dividing the plane into four regions]

top region: $C + S > 0$, $C - S < 0$

left region: $C + S < 0$, $C - S < 0$

right region: $C + S > 0$, $C - S > 0$

bottom region: $C + S < 0$, $C - S > 0$

Ex $\cos 10^\circ - \sin 10^\circ > 0$, $\cos 130^\circ - \sin 130^\circ < 0$

Ex $4\sin 27^\circ = \underline{\hspace{2cm}}$

Sol let $A = 54^\circ$

$$\Rightarrow \frac{A}{2} = 27^\circ \in (-45^\circ, 45^\circ)$$

$$\Rightarrow C + S > 0$$

$$C - S > 0$$

$$\cos(\frac{A}{2}) + \sin(\frac{A}{2}) > 0$$

$$\cos(\frac{A}{2}) - \sin(\frac{A}{2}) > 0$$

$$\sqrt{1 + \sin A} = |\cos(\frac{A}{2}) + \sin(\frac{A}{2})| = \cos(\frac{A}{2}) + \sin(\frac{A}{2}) \text{ --- ①}$$

$$\sqrt{1 - \sin A} = |\cos(\frac{A}{2}) - \sin(\frac{A}{2})| = \cos(\frac{A}{2}) - \sin(\frac{A}{2}) \text{ --- ②}$$

[both expressions inside the bars marked +ve]

$$\sqrt{1 + \sin A} - \sqrt{1 - \sin A} = 2\sin(\frac{A}{2})$$

put $A = 54^\circ$

$$\sqrt{1 + \sin 54^\circ} - \sqrt{1 - \sin 54^\circ} = 2\sin 27^\circ$$

$$\sqrt{1 + \left(\frac{\sqrt{5} + 1}{4}\right)} - \sqrt{1 - \left(\frac{\sqrt{5} + 1}{4}\right)} = 2\sin 27^\circ$$

$$\frac{\sqrt{5 + \sqrt{5}} - \sqrt{3 - \sqrt{5}}}{2} = 2\sin 27^\circ$$

$$\Rightarrow 4\sin 27^\circ = \sqrt{5 + \sqrt{5}} - \sqrt{3 - \sqrt{5}}$$

componendo and dividendo Rule

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$(\text{or}) \frac{a-b}{a+b} = \frac{c-d}{c+d}$$

$$\text{Note: If } \frac{x}{y} = \frac{\sin A}{\sin B} \Rightarrow \frac{x+y}{x-y} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

$$\text{Proof } \frac{x}{y} = \frac{\sin A}{\sin B}$$

$$\begin{aligned} \Rightarrow \frac{x+y}{x-y} &= \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} \\ &= \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} \end{aligned}$$

$$\text{Note If } \frac{x}{y} = \frac{\tan A}{\tan B} \text{ then } \frac{x+y}{x-y} = \frac{\sin(A+B)}{\sin(A-B)}$$

$$\text{Proof } \frac{x}{y} = \frac{\tan A}{\tan B}$$

$$\begin{aligned} \Rightarrow \frac{x+y}{x-y} &= \frac{\tan A + \tan B}{\tan A - \tan B} \\ &= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}} = \frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B} \\ &= \frac{\sin(A+B)}{\sin(A-B)} \end{aligned}$$

Transformations

$$1) \sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$$

$$2) \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Result : In a $\triangle ABC$ i.e $A + B + C = 180^\circ$ (or) π

then (i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right)$

✓ (iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)$

Proof (i) Given $A + B + C = 180^\circ$

$$\begin{aligned}
 \text{LHS} &= \sin 2A + \sin 2B + \sin 2C \\
 &= 2 \sin(A + B) \cos(A - B) + \sin 2C \\
 &= 2 \sin C \cos(A - B) + 2 \sin C \cos C \\
 &= 2 \sin C [\cos(A - B) + \cos C] \\
 &= 2 \sin C [\cos(A - B) - \cos(A + B)] \\
 &= 2 \sin C [2 \sin A \sin B] \\
 &= 4 \sin A \sin B \sin C \\
 &= \text{RHS}
 \end{aligned}$$

Rem

Formula

Replace

Simple

Replace

Formula

$$\begin{aligned}
 &\because \sin C + \sin D \\
 &= 2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)
 \end{aligned}$$

$$\sin(A + B) = \sin(180^\circ - C)$$

$$= \sin C$$

$$\cos C$$

$$= \cos(180^\circ - (A + B))$$

$$= -\cos(A + B)$$

(iv) Given $A + B + C = 180^\circ$

$$\begin{aligned}
 \text{LHS} &= \cos A + \cos B + \cos C \\
 &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) + \cos C \\
 &= 2 \sin\left(\frac{C}{2}\right) \cos\left(\frac{A - B}{2}\right) + 1 - 2 \sin^2\left(\frac{C}{2}\right) \\
 &= 1 + 2 \sin\left(\frac{C}{2}\right) \left[\cos\left(\frac{A - B}{2}\right) - \sin\left(\frac{C}{2}\right)\right] \\
 &= 1 + 2 \sin\left(\frac{C}{2}\right) \left[\cos\left(\frac{A - B}{2}\right) - \cos\left(\frac{A + B}{2}\right)\right]
 \end{aligned}$$

$$= 1 + 2\sin\left(\frac{C}{2}\right)\left[2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right]$$

$$= 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= \text{RHS}$$

$$\cos C + \cos D$$

$$= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos\left(\frac{A+B}{2}\right)$$

$$= \cos\left(\frac{180^\circ - C}{2}\right)$$

$$= \cos\left(90^\circ - \frac{C}{2}\right)$$

$$= \sin\left(\frac{C}{2}\right)$$

$$\cos(A-B) - \cos(A+B)$$

$$= 2\sin A \sin B$$

Ex : P.T $\cos 20^\circ \cos 40^\circ - \sin 5^\circ \sin 25^\circ = \frac{\sqrt{3} + 1}{4}$

Sol : LHS = $\cos 40^\circ \cos 20^\circ - \sin 25^\circ \sin 5^\circ$

$$= \left(\frac{1}{2}\right)[2\cos 40^\circ \cos 20^\circ - 2\sin 25^\circ \sin 5^\circ]$$

$$= \left(\frac{1}{2}\right)[(\cos 60^\circ + \cos 20^\circ) - (\cos 20^\circ - \cos 30^\circ)]$$

$$= \left(\frac{1}{2}\right)\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3} + 1}{2}\right) = \frac{\sqrt{3} + 1}{4} = \text{RHS}$$

$$2\cos A \cos B$$

$$= \cos(A+B) + \cos(A-B)$$

$$2\sin A \sin B$$

$$= \cos(A-B) - \cos(A+B)$$

Ex P.T $4(\cos 66^\circ + \sin 84^\circ) = \sqrt{3} + \sqrt{15}$.

Sol LHS = $4[\cos 66^\circ + \sin 84^\circ]$

$$= 4[\cos 66^\circ + \cos 6^\circ]$$

$$= 4 \cdot 2\cos(36^\circ)\cos(30^\circ)$$

$$= 8\left(\frac{\sqrt{5} + 1}{4}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{15} + \sqrt{3}$$

$$= \text{RHS}$$

$$\sin \theta = \cos(90^\circ - \theta)$$

$$\sin 84^\circ = \cos 6^\circ$$

$$\cos C + \cos D$$

$$= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Ex P.T. $\frac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta} = \tan \theta$

Sol LHS = $\frac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta}$

$$= \frac{(1 - \cos 2\theta) + \sin 2\theta}{(1 + \cos 2\theta) + \sin 2\theta}$$

$$= \frac{2\sin^2 \theta + 2\sin \theta \cos \theta}{2\cos^2 \theta + 2\sin \theta \cos \theta}$$

$$= \frac{2\sin \theta (\sin \theta + \cos \theta)}{2\cos \theta (\cos \theta + \sin \theta)} = \tan \theta = \text{RHS}$$

→ If $\tan \theta = \frac{b}{a}$ then P.T. $a \cos 2\theta + b \sin 2\theta = a$

Sol Given $\tan \theta = \frac{b}{a}$

$$\text{LHS} = a \cos 2\theta + b \sin 2\theta$$

$$= a \left[\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right] + b \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right)$$

$$= a \left[\frac{1 - \frac{b^2}{a^2}}{1 + \frac{b^2}{a^2}} \right] + b \left[\frac{2 \left(\frac{b}{a} \right)}{1 + \frac{b^2}{a^2}} \right]$$

$$= a \left(\frac{a^2 - b^2}{a^2 + b^2} \right) + b \left(\frac{2b}{a} \right) \left(\frac{a^2}{a^2 + b^2} \right)$$

$$= \frac{a(a^2 - b^2) + 2ab^2}{a^2 + b^2}$$

$$= a \left[\frac{a^2 - b^2 + 2b^2}{a^2 + b^2} \right]$$

$$= a \left(\frac{a^2 + b^2}{a^2 + b^2} \right)$$

$$= a = \text{RHS}$$

→ P.T. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$

Sol LHS = $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ}$

$$= \frac{\cos 10^\circ - \sqrt{3} \sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$$

$$= 2 \left[\left(\frac{1}{2} \right) \cos 10^\circ - \left(\frac{\sqrt{3}}{2} \right) \sin 10^\circ \right] / \left(\left(\frac{1}{2} \right) [2 \sin 10^\circ \cos 10^\circ] \right)$$

$$\begin{aligned}
 &= 4[\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ] / \sin 2(10^\circ) \\
 &= 4 \frac{\sin(30^\circ - 10^\circ)}{\sin 20^\circ} \\
 &= 4 \frac{\sin 20^\circ}{\sin 20^\circ} \\
 &= 4 = \text{RHS}
 \end{aligned}$$

$$a \cos \theta + b \sin \theta$$

$$\begin{aligned}
 &\sqrt{a^2 + b^2} \left[\left(\frac{a}{\sqrt{a^2 + b^2}} \right) \cos \theta \right. \\
 &\left. + \left(\frac{b}{\sqrt{a^2 + b^2}} \right) \sin \theta \right]
 \end{aligned}$$

Ex : If $\sin \alpha = \frac{3}{5}$, $\frac{\pi}{2} < \alpha < \pi$, find $\cos 3\alpha$, $\tan 2\alpha$.

Sol Given $\sin \alpha = \frac{3}{5}$, $\alpha \in \text{II}_Q$

[diagram: right triangle with sides 3, 4, hypotenuse 5, angle α opposite side 3]

$$\cos \alpha = -\frac{4}{5}$$

$$\tan \alpha = -\frac{3}{4}$$

$$\cos 3\alpha = 4\cos^3 \alpha - 3\cos \alpha$$

$$= 4\left(-\frac{4}{5}\right)^3 - 3\left(-\frac{4}{5}\right)$$

$$= \left(-\frac{4}{5}\right)\left[4\left(-\frac{4}{5}\right)^2 - 3\right]$$

$$= -\frac{4}{5}\left[\frac{64}{25} - 3\right]$$

$$= -\frac{4}{5}\left(-\frac{11}{25}\right) = \frac{44}{125}$$

$$\tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha}$$

$$= \frac{2\left(-\frac{3}{4}\right)}{1 - \frac{9}{16}}$$

$$= -\frac{3}{2} \times \frac{16}{7}$$

$$= -\frac{24}{7}$$

Ex : Find the value of $\cos 42^\circ + \cos 78^\circ + \cos 162^\circ$

Sol : $\cos 42^\circ + \cos 78^\circ + \cos 162^\circ$

$$= 2\cos\left(\frac{78^\circ + 42^\circ}{2}\right) \cos\left(\frac{78^\circ - 42^\circ}{2}\right)$$

$$+ \cos(180^\circ - 18^\circ)$$

$$\begin{aligned}
 &= 2\cos 60^\circ \cos 18^\circ - \cos 18^\circ \\
 &= 2\left(\frac{1}{2}\right)\cos 18^\circ - \cos 18^\circ \\
 &= \cos 18^\circ - \cos 18^\circ \\
 &= 0
 \end{aligned}$$

$$\cos C + \cos D$$

$$= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Ex P.T $\cos 12^\circ + \cos 132^\circ + \cos 84^\circ + \cos 156^\circ = -\frac{1}{2}$

Sol LHS = $\cos 132^\circ + \cos 12^\circ + \cos 156^\circ + \cos 84^\circ$

$$= 2\cos 72^\circ \cos 60^\circ + 2\cos 120^\circ \cos 36^\circ$$

$$= 2\sin 18^\circ \cdot \left(\frac{1}{2}\right) + 2\left(-\frac{1}{2}\right)\cos 36^\circ$$

$$= \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{\sqrt{5}-1-\sqrt{5}-1}{4}$$

$$= -\frac{2}{4} = -\frac{1}{2} = \text{RHS}$$

$$\cos C + \cos D$$

$$= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Ex : If $x = \frac{2\sin\theta}{1 + \cos\theta + \sin\theta}$ then find the value

of $\frac{1 - \cos\theta + \sin\theta}{1 + \sin\theta}$

Sol Given $x = \frac{2\sin\theta}{1 + \cos\theta + \sin\theta}$

$$= \frac{2\sin\theta (1 - \cos\theta + \sin\theta)}{[1 + \cos\theta + \sin\theta][1 - \cos\theta + \sin\theta]}$$

$$= \frac{2\sin\theta (1 - \cos\theta + \sin\theta)}{(1 + \sin\theta)^2 - \cos^2\theta}$$

$$= \frac{2\sin\theta (1 - \cos\theta + \sin\theta)}{1 + \sin^2\theta + 2\sin\theta - \cos^2\theta}$$

$$= 2\sin\theta[1 - \cos\theta + \sin\theta]/(2\sin^2\theta + 2\sin\theta)$$

$$= 2\sin\theta[1 - \cos\theta + \sin\theta]/(2\sin\theta(1 + \sin\theta))$$

$$x = \frac{1 - \cos\theta + \sin\theta}{1 + \sin\theta}$$

$$\therefore \frac{1 - \cos\theta + \sin\theta}{1 + \sin\theta} = x.$$

Ex : S.T. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta = \sec^2\theta \cdot \csc^2\theta$

Sol $(\tan\theta + \cot\theta)^2$
 $= \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)^2$
 $= \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)^2$
 $= \frac{1}{\sin^2\theta \cdot \cos^2\theta} \quad \text{--- ①}$
 $= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta \cdot \cos^2\theta}$
 $= \frac{\sin^2\theta}{\sin^2\theta \cdot \cos^2\theta} + \frac{\cos^2\theta}{\sin^2\theta \cos^2\theta}$
 $= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$
 $= \sec^2\theta + \csc^2\theta. \quad \text{--- ②}$

From ①

$$\begin{aligned} (\tan\theta + \cot\theta)^2 &= \frac{1}{\sin^2\theta \cos^2\theta} \\ &= \sec^2\theta \cdot \csc^2\theta \end{aligned}$$

Ex : Show that $\cos^4\alpha + 2\cos^2\alpha\left(1 - \frac{1}{\sec^2\alpha}\right) = 1 - \sin^4\alpha$.

Sol LHS = $\cos^4\alpha + 2\cos^2\alpha(1 - \cos^2\alpha)$
 $= \cos^4\alpha + 2\cos^2\alpha(\sin^2\alpha)$
 $= \cos^2\alpha[\cos^2\alpha + 2\sin^2\alpha]$
 $= (1 - \sin^2\alpha)(1 - \sin^2\alpha + 2\sin^2\alpha)$
 $= (1 - \sin^2\alpha)(1 + \sin^2\alpha)$
 $= 1 - \sin^4\alpha$
 $= \text{RHS}$

Ex VIMP

prove that $\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta = \cot\theta$

Sol : $\tan\theta - \cot\theta = \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$
 $= \frac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}$
 $= -[\cos^2\theta - \sin^2\theta] / \left(\frac{1}{2}\right)(2\sin\theta\cos\theta)$
 $= -\frac{2(\cos 2\theta)}{\sin 2\theta}$

$$\tan\theta - \cot\theta = -2\cot 2\theta$$

$$\Rightarrow \tan\theta = \cot\theta - 2\cot 2\theta$$

$$\begin{aligned}
 \text{LHS} &= \tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta \\
 &= (\cot\theta - 2\cot 2\theta) + 2(\cot 2\theta - 2\cot 4\theta) \\
 &+ 4(\cot 4\theta - 2\cot 8\theta) + 8\cot 8\theta \\
 &= \cot\theta - 2\cot 2\theta + 2\cot 2\theta - 4\cot 4\theta + 4\cot 4\theta \\
 &- 8\cot 8\theta + 8\cot 8\theta \\
 &= \cot\theta \\
 &= \text{RHS}
 \end{aligned}$$

Ex If $0 < A < B < \frac{\pi}{4}$, $\sin(A + B) = \frac{24}{25}$, $\cos(A - B) = \frac{4}{5}$

find the value of $\tan 2A$.

Sol $\tan 2A = \tan[(A + B) + (A - B)]$

$$= \frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B)\tan(A - B)} \text{ --- ①}$$

Given $0 < A < B < \frac{\pi}{4}$

$A + B \in I_Q$

$A < B$

$\Rightarrow A - B < 0$

$\tan(A + B) > 0$, $\tan(A - B) < 0$

$\frac{96}{4}$

[diagram: right triangle with sides 24, 7, hypotenuse 25, angle $A + B$]

$\tan(A + B) = \frac{24}{7}$ ✓

[diagram: right triangle with sides 3, 4, hypotenuse 5, angle $A - B$]

$\tan(A - B) = -\frac{3}{4}$ ✓

From ①, $\tan 2A = \frac{\frac{24}{7} - \frac{3}{4}}{1 - \left(\frac{24}{7}\right)\left(-\frac{3}{4}\right)}$

$$= \frac{75}{100} = \frac{3}{4}$$

Ex P.T. $\cos\left(\frac{\pi}{11}\right) \cos\left(\frac{2\pi}{11}\right) \cos\left(\frac{3\pi}{11}\right) \cos\left(\frac{4\pi}{11}\right) \cos\left(\frac{5\pi}{11}\right) = \frac{1}{32}$.

Sol LHS = $\cos\left(\frac{\pi}{11}\right) \cos\left(\frac{2\pi}{11}\right) \cos\left(\frac{4\pi}{11}\right) \cos\left(\frac{8\pi}{11}\right) \cos\left(\frac{16\pi}{11}\right)$

$$= \frac{\sin\left(\frac{32\pi}{11}\right)}{32\sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\sin\left(3\pi - \frac{\pi}{11}\right)}{32\sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{11}\right)}{32\sin\left(\frac{\pi}{11}\right)} = \frac{1}{32} = \text{RHS}$$

$$\cos\theta \cos 2\theta \cos 4\theta = \frac{\sin 8\theta}{8\sin\theta}$$

$$\cos\left(\frac{3\pi}{11}\right)$$

$$= \cos\left(\pi - \frac{8\pi}{11}\right)$$

$$= -\cos\left(\frac{8\pi}{11}\right)$$

$$\cos\left(\frac{5\pi}{11}\right) = \cos\left(\pi - \frac{16\pi}{11}\right)$$

$$= -\cos\left(\frac{16\pi}{11}\right)$$

Ex P.T. $\sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \sin^4\left(\frac{5\pi}{8}\right) + \sin^4\left(\frac{7\pi}{8}\right) = \frac{3}{2}$

Sol : LHS = $\sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{2} - \frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{2} + \frac{\pi}{8}\right) + \sin^4\left(\pi - \frac{\pi}{8}\right)$

$$= \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{8}\right)$$

$$= 2\left(\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)\right)$$

$$= 2\left(1 - 2\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$= 2\left(1 - \left(\frac{1}{2}\right) \cdot 4\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$\sin^4\theta + \cos^4\theta$$

$$= (\sin^2\theta)^2 + (\cos^2\theta)^2$$

$$= (\sin^2\theta + \cos^2\theta)^2$$

$$- 2\sin^2\theta\cos^2\theta$$

$$= 1 - 2\sin^2\theta\cos^2\theta$$

$$\begin{aligned}
 &= 2\left(1 - \left(\frac{1}{2}\right) \cdot \left(2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^2\right) \\
 &= 2\left(1 - \left(\frac{1}{2}\right)\sin^2\left(\frac{\pi}{4}\right)\right) \\
 &= 2\left(1 - \left(\frac{1}{2}\right) \cdot \left(\frac{1}{\sqrt{2}}\right)^2\right) \\
 &= 2\left(1 - \frac{1}{4}\right) \\
 &= 2\left(\frac{3}{4}\right) \\
 &= \frac{3}{2} = \text{RHS}
 \end{aligned}$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

Ex : P.T $(1 + \cos(\frac{\pi}{10}))(1 + \cos(\frac{3\pi}{10}))(1 + \cos(\frac{7\pi}{10}))(1 + \cos(\frac{9\pi}{10})) = \frac{1}{16}$

Sol LHS = $(1 + \cos(\frac{\pi}{10}))(1 + \cos(\frac{3\pi}{10}))(1 - \cos(\frac{3\pi}{10}))(1 - \cos(\frac{\pi}{10}))$

$$= (1 - \cos^2(\frac{\pi}{10}))(1 - \cos^2(\frac{3\pi}{10}))$$

$$= \sin^2(\frac{\pi}{10}) \sin^2(\frac{3\pi}{10})$$

$$= \sin^2 18^\circ \sin^2 54^\circ$$

$$= \sin^2 18^\circ \cos^2 36^\circ$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2 \left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= \left[\frac{5-1}{4 \cdot 4}\right]^2$$

$$= \left(\frac{4}{4(4)}\right)^2$$

$$= \frac{1}{16}$$

$$\cos\left(\frac{7\pi}{10}\right)$$

$$= \cos\left(\pi - \frac{3\pi}{10}\right)$$

$$= -\cos\left(\frac{3\pi}{10}\right)$$

$$\cos\left(\frac{9\pi}{10}\right) = \cos\left(\pi - \frac{\pi}{10}\right)$$

$$= -\cos\left(\frac{\pi}{10}\right)$$

Ex P.T. $\cos\left(\frac{\pi}{9}\right) \cos\left(\frac{2\pi}{9}\right) \cos\left(\frac{3\pi}{9}\right) \cos\left(\frac{4\pi}{9}\right) = \frac{1}{16}$

Sol LHS = $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$
 $= \left(\frac{1}{2}\right) \cdot \cos 20^\circ \cos(60^\circ - 20^\circ) \cos(60^\circ + 20^\circ)$
 $= \left(\frac{1}{2}\right) \cdot \left(\frac{1}{4}\right) \cos 3(20^\circ)$
 $= \left(\frac{1}{8}\right) \cdot \left(\frac{1}{2}\right)$
 $= \frac{1}{16}$
 $\frac{\pi}{9} = \frac{180^\circ}{9}$
 $= 20^\circ$

$$\cos \theta \cos(60^\circ + \theta) \cos(60^\circ - \theta)$$

$$= \left(\frac{1}{4}\right) \cos 3\theta$$

Ex If $\sec(\theta + \alpha) + \sec(\theta - \alpha) = 2\sec\theta$ then

P.T. $\cos\theta = \pm\sqrt{2} \cos\left(\frac{\alpha}{2}\right)$

Sol Given $\sec(\theta + \alpha) + \sec(\theta - \alpha) = 2\sec\theta$
 $\Rightarrow \frac{1}{\cos(\theta + \alpha)} + \frac{1}{\cos(\theta - \alpha)} = \frac{2}{\cos\theta}$
 $\Rightarrow \frac{\cos(\theta - \alpha) + \cos(\theta + \alpha)}{\cos(\theta + \alpha) \cdot \cos(\theta - \alpha)} = \frac{2}{\cos\theta}$
 $\Rightarrow \frac{2\cos\theta\cos\alpha}{\cos^2\theta - \sin^2\alpha} = \frac{2}{\cos\theta}$
 $\Rightarrow \cos^2\theta \cdot \cos\alpha = \cos^2\theta - \sin^2\alpha$
 $\Rightarrow \sin^2\alpha = \cos^2\theta - \cos^2\theta \cdot \cos\alpha$
 $\Rightarrow \sin^2\alpha = \cos^2\theta(1 - \cos\alpha)$
 $\Rightarrow \left(2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right)^2 = \cos^2\theta \cdot 2\sin^2\left(\frac{\alpha}{2}\right)$
 $\Rightarrow 4\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right) = \cos^2\theta \cdot 2\sin^2\left(\frac{\alpha}{2}\right)$
 $\Rightarrow \cos^2\theta = 2\cos^2\left(\frac{\alpha}{2}\right)$
 $\Rightarrow \cos\theta = \pm\sqrt{2}\cos\left(\frac{\alpha}{2}\right)$

Ex : If $\cos x + \cos y = \frac{4}{5}$ and $\cos x - \cos y = \frac{2}{7}$

then P.T $14\tan\left(\frac{x-y}{2}\right) + 5\cot\left(\frac{x+y}{2}\right) = 0$

Sol
$$\frac{\cos x + \cos y}{\cos x - \cos y} = \frac{\frac{4}{5}}{\frac{2}{7}} = \left(\frac{4}{5}\right) \times \left(\frac{7}{2}\right) = \frac{14}{5}$$

$$\Rightarrow \frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} = \frac{14}{5}$$

$$\Rightarrow \frac{\cot\left(\frac{x+y}{2}\right)}{\tan\left(\frac{x-y}{2}\right)} = -\frac{14}{5}$$

$$\Rightarrow 5\cot\left(\frac{x+y}{2}\right)$$

$$= -14\tan\left(\frac{x-y}{2}\right)$$

$$\Rightarrow 14\tan\left(\frac{x-y}{2}\right) + 5\cot\left(\frac{x+y}{2}\right)$$

$$= 0$$

Ex : P.T. $\sin^2(\alpha - 45^\circ) + \sin^2(\alpha + 15^\circ) - \sin^2(\alpha - 15^\circ) = \frac{1}{2}$

Sol LHS =
$$\frac{1 - \cos(2\alpha - 90^\circ)}{2} + \sin(2\alpha) \cdot \sin(30^\circ)$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)\cos(90^\circ - 2\alpha) + \left(\frac{1}{2}\right)\sin 2\alpha$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)\sin 2\alpha + \left(\frac{1}{2}\right)\sin 2\alpha$$

$$= \frac{1}{2}$$

= RHS.

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\sin^2 A - \sin^2 B$$

$$= \sin(A + B) \sin(A - B)$$

Ex : In a $\triangle ABC$, prove that

$$\cos A + \cos B - \cos C = -1 + 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

Sol Given $A + B + C = 180^\circ$

$$\text{LHS} = \cos A + \cos B - \cos C$$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos C$$

$$= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - (1 - 2\sin^2\left(\frac{C}{2}\right))$$

$$\begin{aligned}
 &= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 1 + 2\sin^2\left(\frac{C}{2}\right) \\
 &= -1 + 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{C}{2}\right)\right) \\
 &= -1 + 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right) \\
 &= -1 + 2\sin\left(\frac{C}{2}\right)\left(2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right) \\
 &= -1 + 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \\
 &= \text{RHS} \\
 \cos\left(\frac{A+B}{2}\right) &= \cos\left(90^\circ - \frac{C}{2}\right) \\
 &= \sin\left(\frac{C}{2}\right)
 \end{aligned}$$

Ex In a $\triangle ABC$, P.T

$$\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) - \cos^2\left(\frac{C}{2}\right) = 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

Sol Given $A + B + C = \pi$

$$\begin{aligned}
 \text{LHS} &= \cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) - \cos^2\left(\frac{C}{2}\right) \\
 &= \cos^2\left(\frac{A}{2}\right) + 1 - \sin^2\left(\frac{B}{2}\right) - \cos^2\left(\frac{C}{2}\right) \\
 &= 1 + \left(\cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{B}{2}\right)\right) - \cos^2\left(\frac{C}{2}\right) \\
 &= 1 + \cos\left(\frac{A}{2} + \frac{B}{2}\right)\cos\left(\frac{A}{2} - \frac{B}{2}\right) - \cos^2\left(\frac{C}{2}\right) \\
 &= 1 + \sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2} - \frac{B}{2}\right) - (1 - \sin^2\left(\frac{C}{2}\right)) \\
 &= \sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2} - \frac{B}{2}\right) + \sin^2\left(\frac{C}{2}\right) \\
 &= \sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2} - \frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)\right] \\
 &= \sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A}{2} - \frac{B}{2}\right) + \cos\left(\frac{A}{2} + \frac{B}{2}\right)\right) \\
 &= \sin\left(\frac{C}{2}\right)\left(2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right) \\
 &= 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \\
 &= \text{RHS} \\
 \cos\left(\frac{A}{2} + \frac{B}{2}\right) &= \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)
 \end{aligned}$$

$$= \sin\left(\frac{C}{2}\right)$$

Ex In a $\triangle ABC$, P.T

$$\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) = 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

Sol Given $A + B + C = \pi$

$$\begin{aligned} \text{RHS} &= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right) \\ &= 1 + 2\sin\left(\frac{\pi - C}{4}\right)[2\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)] \\ &= 1 + 2\sin\left(\frac{A + B}{4}\right)\left[\cos\left(\frac{\pi - A}{4} - \frac{\pi - B}{4}\right) - \cos\left(\frac{\pi - A}{4} + \frac{\pi - B}{4}\right)\right] \\ &= 1 + 2\sin\left(\frac{A + B}{4}\right)\left[\cos\left(\frac{B - A}{4}\right) - \cos\left(\frac{\pi}{2} - \left(\frac{A + B}{4}\right)\right)\right] \\ &= 1 + 2\sin\left(\frac{A + B}{4}\right)\left[\cos\left(\frac{A - B}{4}\right) - \sin\left(\frac{A + B}{4}\right)\right] \\ &= 1 + 2\sin\left(\frac{A + B}{4}\right)\cos\left(\frac{A - B}{4}\right) - 2\sin^2\left(\frac{A + B}{4}\right) \\ &= \cos\left[2\left(\frac{A + B}{4}\right)\right] + \sin\left(\frac{A + B}{4} + \frac{A - B}{4}\right) \\ &\quad + \sin\left(\frac{A + B}{4} - \frac{A - B}{4}\right) \\ &= \cos\left(\frac{A + B}{2}\right) + \sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) \\ &= \sin\left(\frac{C}{2}\right) + \sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) \\ &= \sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) \\ &= \text{LHS} \end{aligned}$$

$$1 - 2\sin^2\theta = \cos 2\theta$$

$$2\sin A \cos B$$

$$= \sin(A + B)$$

$$+ \sin(A - B)$$

$$\cos\left(\frac{A + B}{2}\right)$$

$$= \cos\left(90^\circ - \frac{C}{2}\right)$$

$$= \sin\left(\frac{C}{2}\right)$$

Ex : If $A + B + C = 2S$ then prove that

$$\cos(S - A) + \cos(S - B) + \cos(S - C) + \cos S = 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$

Sol Given $A + B + C = 2S$

$$\begin{aligned} \text{LHS} &= \cos(S - A) + \cos(S - B) + \cos(S - C) + \cos S \\ &= 2\cos\left(\frac{S - A + S - B}{2}\right)\cos\left(\frac{S - A - S + B}{2}\right) \\ &\quad + 2\cos\left(\frac{S - C + S}{2}\right)\cos\left(\frac{S - C - S}{2}\right) \\ &= 2\cos\left(\frac{2S - A - B}{2}\right)\cos\left(\frac{B - A}{2}\right) + 2\cos\left(\frac{2S - C}{2}\right)\cos\left(\frac{C}{2}\right) \\ &= 2\cos\left(\frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{C}{2}\right) \\ &= 2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A - B}{2}\right) + \cos\left(\frac{A + B}{2}\right)\right] \\ &= 2\cos\left(\frac{C}{2}\right)\left[2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right] \\ &= 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) \\ &= \text{RHS} \end{aligned}$$

Ex : If $A + B + C = 270^\circ$ then P.T

$$\cos^2 A + \cos^2 B - \cos^2 C = -2\cos A \cos B \sin C$$

Sol Given $A + B + C = 270^\circ$

$$\begin{aligned} \text{LHS} &= \cos^2 A + \cos^2 B - \cos^2 C \\ &= \cos^2 A + 1 - \sin^2 B - \cos^2 C \\ &= 1 + \cos(A + B)\cos(A - B) - \cos^2 C \\ &= 1 - \sin C \cos(A - B) - (1 - \sin^2 C) \\ &= 1 - \sin C \cos(A - B) - 1 + \sin^2 C \\ &= -\sin C [\cos(A - B) - \sin C] \\ &= -\sin C [\cos(A - B) + \cos(A + B)] \\ &= -\sin C [2\cos A \cos B] \\ &= -2\cos A \cos B \sin C \\ &= \text{RHS} \end{aligned}$$

$$\begin{aligned} \cos(A + B) &= \cos(270^\circ - C) \\ &= -\sin C \end{aligned}$$

$$\rightarrow \text{If } \frac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - m}{1 + m} \text{ then P.T}$$

$$\tan\left(\frac{\pi}{4} - \alpha\right) = m \tan\left(\frac{\pi}{4} + \beta\right)$$

$$\text{Sol Given } \frac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - m}{1 + m}$$

$$\Rightarrow \frac{\sin(\alpha + \beta) + \cos(\alpha - \beta)}{\sin(\alpha + \beta) - \cos(\alpha - \beta)} = \frac{(1 - m) + (1 + m)}{(1 - m) - (1 + m)}$$

$$\Rightarrow \frac{\sin(\alpha + \beta) + \sin\left[\frac{\pi}{2} - \alpha + \beta\right]}{\sin(\alpha + \beta) - \sin\left[\frac{\pi}{2} - \alpha + \beta\right]} = \frac{1 - m + 1 + m}{1 - m - 1 - m} = \frac{2}{-2m} = -\frac{1}{m}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow \frac{a + b}{a - b} = \frac{c + d}{c - d}$$

$$\Rightarrow \frac{2\sin\left(\frac{\pi}{4} + \beta\right)\cos\left(\alpha - \frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{4} + \beta\right)\sin\left(\alpha - \frac{\pi}{4}\right)} = -\frac{1}{m}$$

$$\Rightarrow \frac{\tan\left(\frac{\pi}{4} + \beta\right)}{\tan\left(\alpha - \frac{\pi}{4}\right)} = -\frac{1}{m}$$

$$\Rightarrow \frac{\tan\left(\frac{\pi}{4} + \beta\right)}{-\tan\left(\frac{\pi}{4} - \alpha\right)} = -\frac{1}{m}$$

$$\Rightarrow m \tan\left(\frac{\pi}{4} + \beta\right) = \tan\left(\frac{\pi}{4} - \alpha\right)$$

$$\therefore \tan\left(\frac{\pi}{4} - \alpha\right) = m \tan\left(\frac{\pi}{4} + \beta\right)$$

$$\tan(-\theta) = -\tan\theta$$

periodicity and extreme values

$$\text{let } f(x) = \sin x$$

$$f(2\pi + x) = \sin(2\pi + x) = \sin x = f(x)$$

$$f(4\pi + x) = f(x)$$

$$f(6\pi + x) = f(x)$$

where 2π is the least +ve

real number such that $f(2\pi + x) = f(x)$

$$\therefore \text{period} = 2\pi.$$

Def : A function $f: A \rightarrow R$ is said to be a periodic function with period T if

$$f(x + T) = f(x), \forall x \in A.$$

The least +ve real value of T

such that $f(x + T) = f(x)$ is called period of f .

Note [table]

Function | period

$$\sin x \mid 2\pi$$

$$\cos x \mid 2\pi$$

$$\tan x \mid \pi$$

$$\cot x \mid \pi$$

$$\sec x \mid 2\pi$$

$$\csc x \mid 2\pi$$

$$\tan(\pi + x) = \tan x$$

$$\tan(2\pi + x) = \tan x$$

Note If f is a periodic function with period T then $-f$ and $\frac{1}{f}$ are also periodic functions with period T

$$f(x) = \sin x, \text{ period} = 2\pi.$$

Consider

$$g(x) = -\sin x$$

$$g(x + 2\pi) = -\sin(x + 2\pi)$$

$$= -\sin x = g(x)$$

$$\therefore \text{period of } -\sin x$$

$$= 2\pi.$$

Note If the period of $f(x)$ is p

then the period of $f(ax + b) + k$ is $p/|a|$

Ex [table]

Function | period

$$\sin(ax + b) \mid 2\pi/|a|$$

$$\cos(ax + b) \mid 2\pi/|a|$$

$$\tan(ax + b) \mid \pi/|a|$$

$$\cot(ax + b) \mid \pi/|a|$$

$$\sec(ax + b) \mid 2\pi/|a|$$

$$\csc(ax + b) \mid 2\pi/|a|$$

Ex period of $2\sin(3x + 5) + 8$ is $\frac{2\pi}{3}$

Ex period of $\tan(5 - 3x)$ is $\frac{\pi}{3}$

Note If $f(x)$ and $g(x)$ are two functions whose periods p_1 and p_2 and there exists

no positive real α such that $f(x + \alpha) = g(x)$,

$g(x + \alpha) = f(x)$ then the period of $f(x) \pm g(x)$

(or) $\frac{f(x)}{g(x)}$ is the L.C.M of $\{p_1, p_2\}$

Ex period of $\sin(\frac{x}{2}) + \cos(\frac{x}{3})$ is ____

Sol period = L.C.M $\{\frac{2\pi}{2}, \frac{2\pi}{3}\}$

$$= \text{L.C.M}\{4\pi, 6\pi\}$$

$$= 12\pi$$

Ex period of $\sin^2 x$ is ____

Sol $f(x) = \sin^2 x$

$$= \frac{1 - \cos 2x}{2} = \frac{1}{2} - \left(\frac{1}{2}\right)\cos 2x$$

$$\text{period} = \frac{2\pi}{2} = \pi.$$

$$2 \mid 4, 6$$

$$2, 3$$

$$2 \times 2 \times 3 = 12$$

Ind method

$$\text{Let } f(x) = \sin^2 x$$

$$f(x + 2\pi) = \sin^2(x + 2\pi) = \sin^2 x = f(x)$$

$$f(x + \pi) = \sin^2(x + \pi) = (-\sin x)^2 = \sin^2 x = f(x)$$

where π is least +ve real

such that $f(x + \pi) = f(x)$

$$\therefore \text{period of } \sin^2 x = \pi.$$

Ex period of $\sin^3 x$.

Sol $\sin 3x = 3\sin x - 4\sin^3 x$.

$$4\sin^3 x = 3\sin x - \sin 3x$$

$$\sin^3 x = \left(\frac{3}{4}\right)\sin x - \left(\frac{1}{4}\right)\sin 3x.$$

$$\text{period} = \text{L.C.M}\left\{\frac{2\pi}{1}, \frac{2\pi}{3}\right\}$$

$$= \text{L.C.M}\{2\pi, 2\pi\} / \text{G.C.D}\{1, 3\}$$

$$= \frac{2\pi}{1}$$

$$= 2\pi.$$

$$\text{L.C.M of } \left\{\frac{a}{b}, \frac{c}{d}\right\}$$

$$= \text{L.C.M}\{a, c\} / \text{G.C.D}\{b, d\}$$

Ex period of $\sin^4 x$ is ____

Sol $f(x) = \sin^4 x$

$$f(x + \pi) = \sin^4(x + \pi) = (-\sin x)^4 = \sin^4 x = f(x)$$

$$f(x + 2\pi) = f(x)$$

$$f(x + 4\pi) = f(x)$$

where π is the least +ve

real no. such that $f(x + \pi) = f(x)$

$\therefore$ period = π .

Note : [table]

Function | period

$\sin^n x$ (n is even) | π

$\cos^n x$ (n is even) | π .

$\tan^n x$ (n is even) | π ✓

$\sin^n x, \cos^n x$ (n is odd) | 2π

$\tan^n x$ (n is odd) | π . ✓

Note : The period of $|\sin x|, |\cos x|, |\tan x|, |\cot x|,$

$|\sec x|, |\csc x|$ is π

Proof let $f(x) = |\sin x|$

$$f(x + \pi) = |\sin(x + \pi)|$$

$$= |-\sin x|$$

$$= |\sin x|$$

$$= f(x)$$

where π is least +ve real no.

$\therefore$ period = π .

$$|-x| = |x|$$

Note : The period of

$a|\sin x| + b|\cos x|, a|\tan x| + b|\cot x|,$

$a|\csc x| + b|\sec x|$ is π if $a \neq b$

$\frac{\pi}{2}$. if $a = b$

Ex $f(x) = |\sin x| + |\cos x|$

$$f\left(x + \frac{\pi}{2}\right) = |\cos x| + |-\sin x|$$

$$= |\cos x| + |\sin x|$$

$$= f(x)$$

where $\frac{\pi}{2}$ is the least +ve real no

$$\therefore \text{period} = \frac{\pi}{2}.$$