EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Let $P(n)$: $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n+5)}{2}$. Verify that $P(3)$ is true. [1 mark]

2. In proving $3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 + \cdots + (n + 2)(n + 3) = \frac{n(n^2+9n+26)}{3}$ by induction, what term is added to both sides of $P(k)$? [1 mark]

3. Prove by mathematical induction that $1 \cdot 5 + 3 \cdot 7 + 5 \cdot 9 + \cdots + (2n - 1)(2n + 3) = \frac{n(4n^2+12n-1)}{3}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5n-3)(5n+2)} = \frac{n}{10n+4}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $1^2 + 2^2 + 3^2 + \cdots + 30^2$. [1 mark]

6. Find $1^4 + 2^4 + 3^4 + \cdots + 8^4$ using the formula. [2 marks]

7. Find the sum $1^2 + 3^2 + 5^2 + \ldots$ to 12 terms. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 25^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} (2k - 1)(3k + 1)$ in factorised form. [3 marks]

10. Find the sum of the first n terms of $1 + 7 + 13 + \ldots$ [2 marks]

11. Find the sum of the first n terms of $7 + 14 + 28 + \ldots$ [2 marks]

12. Find the sum to infinity of $5 + 7 \cdot \frac{1}{4} + 9 \cdot \frac{1}{16} + 11 \cdot \frac{1}{64} + \ldots$ [3 marks]

13. Find the nth term of the sequence $2, 11, 28, 53, \ldots$ [3 marks]

14. Find the nth term of the sequence $5, 29, 125, 509, \ldots$ [3 marks]

15. A series is written $2^2 + 3^2 + \cdots + (n - 1)^2$, that is, up to $(n - 2)$ terms. With which value of n should the verification begin? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $13^n - 6^n$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $10^n - 9n - 1$ is divisible by 81 for all $n \in \mathbb{N}$. [5 marks]

19. Is $44 \equiv 9 \pmod 7$? Give the reason. [1 mark]

20. 15 is odd. Using a standard result, show that $15(15^2 - 1)$ is divisible by 24 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 21, so $P(3)$ is true
2	$(k+3)(k+4)$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	9455
6	8772
7	2300
8	325
9	$\frac{n(4n^2+5n-1)}{2}$
10	$n(3n-2)$
11	$7 \cdot 2^n - 7$
12	$\frac{68}{9}$
13	$T_n = 4n^2 - 3n + 1$
14	$T_n = 2 \cdot 4^n - 3$
15	$n = 3$
16	7
17	Proved: divisible by 133 for all $n \in \mathbb{N}$
18	Proved: divisible by 81 for all $n \in \mathbb{N}$
19	Yes: $44 - 9 = 35$ is divisible by 7
20	140

Solutions

1. Let $P(n)$: $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n+5)}{2}$. Verify that $P(3)$ is true. [1]

$P(3)$: L.H.S. $= 4 + 7 + 10 = 21$
R.H.S. at $n = 3$: $\frac{n(3n+5)}{2} = 21$
L.H.S. $=$ R.H.S., so $P(3)$ is true.

Ans: Both sides equal 21, so $P(3)$ is true

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2. In proving $3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 + \cdots + (n + 2)(n + 3) = \frac{n(n^2+9n+26)}{3}$ by induction, what term is added to both sides of $P(k)$? [1]

The nth term is $(n + 2)(n + 3)$.
Put $n = k + 1$: the $(k + 1)$th term is $(k + 3)(k + 4)$.

Ans: $(k + 3)(k + 4)$

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3. Prove by mathematical induction that $1 \cdot 5 + 3 \cdot 7 + 5 \cdot 9 + \cdots + (2n - 1)(2n + 3) = \frac{n(4n^2+12n-1)}{3}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $1 \cdot 5 + 3 \cdot 7 + 5 \cdot 9 + \cdots + (2n - 1)(2n + 3) = \frac{n(4n^2+12n-1)}{3}$.
For $n = 1$: L.H.S. $= 1 \cdot 5 = 5$, R.H.S. $= 5$. So $P(1)$ is true.
Assume $P(k)$ is true: $1 \cdot 5 + 3 \cdot 7 + 5 \cdot 9 + \cdots + (2k - 1)(2k + 3) = \frac{k(4k^2+12k-1)}{3}$.
Add the $(k + 1)$th term $(2k + 1)(2k + 5)$ to both sides.
L.H.S. of $P(k + 1) = \frac{k(4k^2+12k-1)}{3} + (2k + 1)(2k + 5)$
$= \frac{4k^3}{3} + 8k^2 + \frac{35k}{3} + 5$
$= \frac{(k+1)(4k^2+20k+15)}{3}$, which is $\frac{n(4n^2+12n-1)}{3}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.
So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5n-3)(5n+2)} = \frac{n}{10n+4}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5n-3)(5n+2)} = \frac{n}{10n+4}$.
For $n = 1$: L.H.S. $= \frac{1}{2 \cdot 7} = \frac{1}{14}$, R.H.S. $= \frac{1}{14}$. So $P(1)$ is true.
Assume $P(k)$ is true: $\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5k-3)(5k+2)} = \frac{k}{10k+4}$.
Add the $(k + 1)$th term $\frac{1}{(5k+2)(5k+7)}$ to both sides.
L.H.S. of $P(k + 1) = \frac{k}{10k+4} + \frac{1}{(5k+2)(5k+7)}$
$= \frac{5k^2+7k+2}{2(5k+2)(5k+7)}$
$= \frac{k+1}{2(5k+7)}$, which is $\frac{n}{10n+4}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.
So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $1^2 + 2^2 + 3^2 + \cdots + 30^2$. [1]

$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
$n = 30$: $\frac{30 \cdot 31 \cdot 61}{6} = 9455$

Ans: 9455

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6. Find $1^4 + 2^4 + 3^4 + \cdots + 8^4$ using the formula. [2]

$\sum n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

$n = 8: \ \frac{8 \cdot 9 \cdot 17 \cdot 215}{30}$

$= 8772$

Ans: 8772

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7. Find the sum $1^2 + 3^2 + 5^2 + \ldots$ to 12 terms. [2]

$\sum (2k-1)^2 = \frac{n}{3}(4n^2 - 1)$

$n = 12: \ \frac{12}{3}(4 \cdot 144 - 1) = \frac{12 \cdot 575}{3} = 2300$

Ans: 2300

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 25^2$. [1]

There are $n = 25$ terms and n is odd.

$\frac{n(n+1)}{2} = \frac{25 \cdot 26}{2} = 325$

Ans: 325

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9. Using the standard sums, find $\sum_{k=1}^{n} (2k-1)(3k+1)$ in factorised form. [3]

$(2k-1)(3k+1) = 6k^2 - k - 1$

$\sum_{k=1}^{n} (6k^2 - k - 1) = 6\sum k^2 - \sum k - \sum 1$

$= 6 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} - n$

$= \frac{4n^3 + 5n^2 - n}{2}$

$= \frac{n(4n^2 + 5n - 1)}{2}$

Ans: $\frac{n(4n^2 + 5n - 1)}{2}$

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10. Find the sum of the first n terms of $1 + 7 + 13 + \ldots$ [2]

$a = 1$, $d = 6$.

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[2 + 6(n-1)]$

$= n(3n - 2)$

Ans: $n(3n - 2)$

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11. Find the sum of the first n terms of $7 + 14 + 28 + \ldots$ [2]

$a = 7$, $r = 2$.

$S_n = a\frac{r^n - 1}{r - 1} = 7 \cdot 2^n - 7$

Ans: $7 \cdot 2^n - 7$

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12. Find the sum to infinity of $5 + 7 \cdot \frac{1}{4} + 9 \cdot \frac{1}{16} + 11 \cdot \frac{1}{64} + \ldots$ [3]

This is an A.G.P. with $a = 5$, $d = 2$, $r = \frac{1}{4}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{5}{\frac{3}{4}} + \frac{\frac{1}{2}}{\frac{9}{16}}$

$= \frac{20}{3} + \frac{8}{9} = \frac{68}{9}$

Ans: $\frac{68}{9}$

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13. Find the nth term of the sequence $2, 11, 28, 53, \ldots$ [3]

Differences: $9, 17, 25$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 2,\ 4a + 2b + c = 11,\ 9a + 3b + c = 28$

$3a + b = 9,\ 5a + b = 17 \Rightarrow a = 4,\ b = -3,\ c = 1$

$\therefore T_n = 4n^2 - 3n + 1$ (check: $T_4 = 53$)

Ans: $T_n = 4n^2 - 3n + 1$

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14. Find the nth term of the sequence $5, 29, 125, 509, \ldots$ [3]

Differences: $24, 96, 384$, in G.P. with ratio 4, so $T_n = A \cdot 4^n + B$.

$n = 1:\ 4A + B = 5;\ n = 2:\ 16A + B = 29$

$12A = 24 \Rightarrow A = 2,\ B = -3$

$\therefore T_n = 2 \cdot 4^n - 3$

Ans: $T_n = 2 \cdot 4^n - 3$

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15. A series is written $2^2 + 3^2 + \cdots + (n-1)^2$, that is, up to $(n-2)$ terms. With which value of n should the verification begin? [1]

The series has $(n-2)$ terms.

It has at least one term only when $(n-2) \geq 1$, that is $n \geq 3$.

So the verification begins with $n = 3$.

Ans: $n = 3$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $13^n - 6^n$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 7$

$n = 2:\ 133$

$\text{G.C.D.}(7, 133) = 7$

So $13^n - 6^n$ is divisible by 7 (induction proves it for all n).

Ans: 7

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17. Prove by mathematical induction that $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $11^{n+2} + 12^{2n+1}$ is divisible by 133.

For $n = 1$: the value is $3059 = 133 \times 23$. So $P(1)$ is true.

Assume $P(k)$: $11^{k+2} + 12^{2k+1} = 133\lambda$ for some integer λ.

Then $11^{k+2} = 133\lambda - 12^{2k+1}$.

At $n = k + 1$: $11^{k+3} + 12^{2k+3} = 11(11^{k+2}) + 144 \cdot 12^{2k+1}$

$= 11(133\lambda - 12^{2k+1}) + 144 \cdot 12^{2k+1}$

$= 1463\lambda + 133 \cdot 12^{2k+1}$

$= 133\left(11\lambda + 12^{2k+1}\right)$, which is divisible by 133.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 133 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $10^n - 9n - 1$ is divisible by 81 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $10^n - 9n - 1$ is divisible by 81.

For $n = 1$: $10 - 9 - 1 = 0$, which is divisible by 81. So $P(1)$ is true.

Assume $P(k)$: $10^k - 9k - 1 = 81\lambda$, so $10^k = 81\lambda + 9k + 1$.

At $n = k + 1$: $10^{k+1} - 9(k+1) - 1 = 10 \cdot 10^k - 9k - 10$

$= 10(81\lambda + 9k + 1) - 9k - 10$

$= 810\lambda + 90k + 10 - 9k - 10$

$= 810\lambda + 81k = 81(10\lambda + k)$, which is divisible by 81.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 81 for all $n \in \mathbb{N}$

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19. Is $44 \equiv 9 \pmod 7$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$44 - 9 = 35$

$35 = 7 \times 5$, so yes.

Ans: Yes: $44 - 9 = 35$ is divisible by 7

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20. 15 is odd. Using a standard result, show that $15(15^2 - 1)$ is divisible by 24 and find the quotient. [2]

For odd n, $n(n^2 - 1)$ is divisible by 24.

$15(225 - 1) = 15 \times 224 = 3360 = 24 \times 140$

Ans: 140
