EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 08

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For the statement $P(n)$: $3 + 10 + 17 + \cdots + (7n - 4) = \frac{n(7n-1)}{2}$, check $P(4)$. [1 mark]

2. To prove $1 + 6 + 36 + \cdots + 6^{n-1} = \frac{6^n - 1}{5}$ by induction, which term is added to both sides of $P(k)$ to get $P(k+1)$? [1 mark]

3. Prove by mathematical induction that $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \cdots + (3n-1)(3n+2) = n(3n^2 + 6n + 1)$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $4 + 16 + 64 + \cdots + 4 \cdot 4^{n-1} = \frac{4(4^n - 1)}{3}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $\sum_{k=1}^{75} k$. [1 mark]

6. Find $16^2 + 17^2 + \cdots + 30^2$. [2 marks]

7. Find $2 + 4 + 6 + \cdots + 120$. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 14^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} k(k+1)(2k+1)$ in factorised form. [3 marks]

10. Find the sum $3 + 11 + 19 + \cdots + 107$. [2 marks]

11. Find the sum to infinity of $50 + 10 + 2 + \frac{2}{5} + \ldots$ [2 marks]

12. Using the A.G.P. formula, find $1 + 3 \cdot 3 + 5 \cdot 9 + 7 \cdot 27 + \ldots$ to 5 terms. [3 marks]

13. Find the nth term of the sequence $5, 13, 25, 41, \ldots$ [3 marks]

14. Find the nth term of the sequence $13, 37, 109, 325, \ldots$ [3 marks]

15. In proving $5 + 16 + \cdots + (11n - 39)$ (up to $(n-3)$ terms) is equal to a formula, at which value of n do we start? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $4^{3n} - 1$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $7^{2n} - 1$ is divisible by 48 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $n^3 - 7n + 3$ is divisible by 3 for all $n \in \mathbb{N}$. [5 marks]

19. Is $95 \equiv 18 \pmod{11}$? Give the reason. [1 mark]

20. Using Fermat's result for the prime 7, show that $2^7 - 2$ is divisible by 7 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 54, so $P(4)$ is true
2	6^k
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	2850
6	8215
7	3660
8	-105
9	$\frac{n(n+1)^2(n+2)}{2}$
10	770
11	$\frac{125}{2}$
12	973
13	$T_n = 2n^2 + 2n + 1$
14	$T_n = 4 \cdot 3^n + 1$
15	$n = 4$
16	63
17	Proved: divisible by 48 for all $n \in \mathbb{N}$
18	Proved: divisible by 3 for all $n \in \mathbb{N}$
19	Yes: $95 - 18 = 77$ is divisible by 11
20	18

Solutions

1. For the statement $P(n)$: $3 + 10 + 17 + \cdots + (7n - 4) = \frac{n(7n-1)}{2}$, check $P(4)$. [1]

$P(4)$: L.H.S. $= 3 + 10 + 17 + 24 = 54$

R.H.S. at $n = 4$: $\frac{n(7n-1)}{2} = 54$

L.H.S. $=$ R.H.S., so $P(4)$ is true.

Ans: Both sides equal 54, so $P(4)$ is true

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2. To prove $1 + 6 + 36 + \cdots + 6^{n-1} = \frac{6^n - 1}{5}$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1]

The nth term is 6^{n-1}.

Put $n = k + 1$: the $(k + 1)$th term is 6^k.

Ans: 6^k

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3. Prove by mathematical induction that $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \cdots + (3n - 1)(3n + 2) = n(3n^2 + 6n + 1)$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \cdots + (3n - 1)(3n + 2) = n(3n^2 + 6n + 1)$.

For $n = 1$: L.H.S. $= 2 \cdot 5 = 10$, R.H.S. $= 10$. So $P(1)$ is true.

Assume $P(k)$ is true: $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \cdots + (3k - 1)(3k + 2) = k(3k^2 + 6k + 1)$.

Add the $(k + 1)$th term $(3k + 2)(3k + 5)$ to both sides.

L.H.S. of $P(k + 1) = k(3k^2 + 6k + 1) + (3k + 2)(3k + 5)$

$= 3k^3 + 15k^2 + 22k + 10$

$= (k + 1)(3k^2 + 12k + 10)$, which is $n(3n^2 + 6n + 1)$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $4 + 16 + 64 + \cdots + 4 \cdot 4^{n-1} = \frac{4(4^n - 1)}{3}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $4 + 16 + 64 + \cdots + 4 \cdot 4^{n-1} = \frac{4(4^n - 1)}{3}$.

For $n = 1$: L.H.S. $= 4 = 4$, R.H.S. $= 4$. So $P(1)$ is true.

Assume $P(k)$ is true: $4 + 16 + 64 + \cdots + 4 \cdot 4^{k-1} = \frac{4 \cdot 4^k}{3} - \frac{4}{3}$.

Add the $(k + 1)$th term $4 \cdot 4^k$ to both sides.

L.H.S. of $P(k + 1) = \frac{4 \cdot 4^k}{3} - \frac{4}{3} + 4 \cdot 4^k$

$= \frac{16 \cdot 4^k}{3} - \frac{4}{3}$

$= \frac{4(4 \cdot 4^k - 1)}{3}$, which is $\frac{4(4^n - 1)}{3}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $\sum_{k=1}^{75} k$. [1]

$\sum n = \frac{n(n+1)}{2}$

$n = 75$: $\frac{75 \cdot 76}{2} = 2850$

Ans: 2850

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6. Find $16^2 + 17^2 + \cdots + 30^2$. [2]

The sum from 1 to 30 minus the sum from 1 to 15.

$\sum_{k=1}^{30} k^2 = 9455,\ \sum_{k=1}^{15} k^2 = 1240$

$\therefore \text{sum} = 9455 - 1240 = 8215$

Ans: 8215

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7. Find $2 + 4 + 6 + \cdots + 120$. [2]

$2n = 120 \Rightarrow n = 60$

$2 + 4 + \cdots + 2n = n(n+1) = 60 \cdot 61 = 3660$

Ans: 3660

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 14^2$. [1]

There are $n = 14$ terms and n is even.

$-\frac{n(n+1)}{2} = -\frac{14 \cdot 15}{2} = -105$

Ans: -105

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9. Using the standard sums, find $\sum_{k=1}^{n} k(k+1)(2k+1)$ in factorised form. [3]

$k(k+1)(2k+1) = 2k^3 + 3k^2 + k$

$\sum_{k=1}^{n}(2k^3 + 3k^2 + k) = 2\sum k^3 + 3\sum k^2 + \sum k$

$= 2 \cdot \frac{n^2(n+1)^2}{4} + 3 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n^4 + 4n^3 + 5n^2 + 2n}{2}$

$= \frac{n(n+1)^2(n+2)}{2}$

Ans: $\frac{n(n+1)^2(n+2)}{2}$

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10. Find the sum $3 + 11 + 19 + \cdots + 107$. [2]

$a = 3,\ d = 8,\ l = 107.$

$107 = 3 + (n-1)8 \Rightarrow n = 14$

$S_n = \frac{n}{2}[a + l] = \frac{14}{2}(3 + 107) = 770$

Ans: 770

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11. Find the sum to infinity of $50 + 10 + 2 + \frac{2}{5} + \ldots$ [2]

$a = 50,\ r = \frac{1}{5},\ |r| < 1.$

$S_\infty = \frac{a}{1-r} = \frac{50}{1-(\frac{1}{5})} = \frac{125}{2}$

Ans: $\frac{125}{2}$

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12. Using the A.G.P. formula, find $1 + 3 \cdot 3 + 5 \cdot 9 + 7 \cdot 27 + \ldots$ to 5 terms. [3]

$a = 1,\ d = 2,\ r = 3,\ n = 5$; the last term has $a + (n-1)d = 9$.

$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$

$= \frac{1}{-2} + \frac{6(1-81)}{4} - \frac{9 \cdot 243}{-2}$

$= 973$

Ans: 973

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13. Find the nth term of the sequence $5, 13, 25, 41, \ldots$ [3]

Differences: $8, 12, 16$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 5,\ 4a + 2b + c = 13,\ 9a + 3b + c = 25$

$3a + b = 8,\ 5a + b = 12 \Rightarrow a = 2,\ b = 2,\ c = 1$

$\therefore T_n = 2n^2 + 2n + 1$ (check: $T_4 = 41$)

Ans: $T_n = 2n^2 + 2n + 1$

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14. Find the nth term of the sequence $13, 37, 109, 325, \ldots$ [3]

Differences: $24, 72, 216$, in G.P. with ratio 3, so $T_n = A \cdot 3^n + B$.

$n = 1:\ 3A + B = 13;\ n = 2:\ 9A + B = 37$

$6A = 24 \Rightarrow A = 4,\ B = 1$

$\therefore T_n = 4 \cdot 3^n + 1$

Ans: $T_n = 4 \cdot 3^n + 1$

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15. In proving $5 + 16 + \cdots + (11n - 39)$ (up to $(n - 3)$ terms) is equal to a formula, at which value of n do we start? [1]

The series has $(n - 3)$ terms.

It has at least one term only when $(n - 3) \geq 1$, that is $n \geq 4$.

So the verification begins with $n = 4$.

Ans: $n = 4$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $4^{3n} - 1$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 63$

$n = 2:\ 4095$

$\text{G.C.D.}(63, 4095) = 63$

So $4^{3n} - 1$ is divisible by 63 (induction proves it for all n).

Ans: 63

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17. Prove by mathematical induction that $7^{2n} - 1$ is divisible by 48 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $7^{2n} - 1$ is divisible by 48.

For $n = 1$: the value is $48 = 48 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $7^{2k} - 1 = 48\lambda$ for some integer λ.

Then $7^{2k} = 48\lambda + 1$.

At $n = k + 1$: $7^{2k+2} - 1 = 49(7^{2k}) - 1$

$= 49(48\lambda + 1) - 1$

$= 2352\lambda + 48$

$= 48\,(49\lambda + 1)$, which is divisible by 48.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 48 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $n^3 - 7n + 3$ is divisible by 3 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $n^3 - 7n + 3$ is divisible by 3.

For $n = 1$: the value is -3, divisible by 3. So $P(1)$ is true.

Assume $P(k)$: $k^3 - 7k + 3 = 3\lambda$.

At $n = k + 1$: $(k+1)^3 - 7(k+1) + 3$

$= (k^3 - 7k + 3) + (3k^2 + 3k - 6)$

$= 3\lambda + 3(k^2 + k - 2)$, which is divisible by 3.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 3 for all $n \in \mathbb{N}$

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19. Is $95 \equiv 18 \pmod{11}$? Give the reason. [1]

$x \equiv y \pmod{m}$ means $x - y$ is divisible by m.

$95 - 18 = 77$

$77 = 11 \times 7$, so yes.

Ans: Yes: $95 - 18 = 77$ is divisible by 11

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20. Using Fermat's result for the prime 7, show that $2^7 - 2$ is divisible by 7 and find the quotient. [2]

For a prime p, $n^p - n$ is divisible by p.

$2^7 - 2 = 128 - 2 = 126 = 7 \times 18$

Ans: 18
