EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Let $P(n)$: $5 + 11 + 17 + \cdots + (6n - 1) = n(3n + 2)$. Verify that $P(3)$ is true. [1 mark]

2. In proving $3^2 + 7^2 + 11^2 + \cdots + (4n - 1)^2 = \frac{n(16n^2+12n-1)}{3}$ by induction, what term is added to both sides of $P(k)$? [1 mark]

3. Prove by mathematical induction that $2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + (n + 1)(n + 3) = \frac{n(2n^2+15n+31)}{6}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \frac{1}{9\cdot13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $1^2 + 2^2 + 3^2 + \cdots + 24^2$. [1 mark]

6. Find $1^3 + 2^3 + 3^3 + \cdots + 18^3$. [2 marks]

7. Find $1^3 + 3^3 + 5^3 + \ldots$ to 7 terms. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 19^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} k(k - 1)(k + 1)$ in factorised form. [3 marks]

10. Find the 22th term and the sum of the first 22 terms of $9, 13, 17, \ldots$ [2 marks]

11. Find the 7th term and the sum of the first 7 terms of $1, 4, 16, \ldots$ [2 marks]

12. Find the sum to infinity of $4 + 5 \cdot \frac{1}{3} + 6 \cdot \frac{1}{9} + 7 \cdot \frac{1}{27} + \ldots$ [3 marks]

13. Find the nth term of the sequence $5, 7, 11, 17, \ldots$ [3 marks]

14. Find the nth term of the sequence $3, 23, 123, 623, \ldots$ [3 marks]

15. A series is written $1 \cdot 5 + 5 \cdot 9 + \cdots + (4n - 7)(4n - 3)$, that is, up to $(n - 1)$ terms. With which value of n should the verification begin? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $3 \cdot 4^n + 2^n - 2$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $10^n - 3^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $9^n - 8n - 1$ is divisible by 64 for all $n \in \mathbb{N}$. [5 marks]

19. Is $52 \equiv 7 \pmod 5$? Give the reason. [1 mark]

20. Using a standard result, show that $5 \cdot 6 \cdot 7$ is divisible by $3!$ and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 33, so $P(3)$ is true
2	$(4k+3)^2$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	4900
6	29241
7	4753
8	190
9	$\frac{n(n-1)(n+1)(n+2)}{4}$
10	$t_{22} = 93$, $S_{22} = 1122$
11	$t_7 = 4096$, $S_7 = 5461$
12	$\frac{27}{4}$
13	$T_n = n^2 - n + 5$
14	$T_n = 5^n - 2$
15	$n = 2$
16	2
17	Proved: divisible by 7 for all $n \in \mathbb{N}$
18	Proved: divisible by 64 for all $n \in \mathbb{N}$
19	Yes: $52 - 7 = 45$ is divisible by 5
20	35

Solutions

1. Let $P(n)$: $5 + 11 + 17 + \cdots + (6n - 1) = n(3n + 2)$. Verify that $P(3)$ is true. [1]

$P(3)$: L.H.S. $= 5 + 11 + 17 = 33$

R.H.S. at $n = 3$: $n(3n + 2) = 33$

L.H.S. $=$ R.H.S., so $P(3)$ is true.

Ans: Both sides equal 33, so $P(3)$ is true

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2. In proving $3^2 + 7^2 + 11^2 + \cdots + (4n - 1)^2 = \frac{n(16n^2+12n-1)}{3}$ by induction, what term is added to both sides of $P(k)$? [1]

The nth term is $(4n - 1)^2$.

Put $n = k + 1$: the $(k + 1)$th term is $(4k + 3)^2$.

Ans: $(4k + 3)^2$

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3. Prove by mathematical induction that $2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + (n + 1)(n + 3) = \frac{n(2n^2+15n+31)}{6}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + (n + 1)(n + 3) = \frac{n(2n^2+15n+31)}{6}$.

For $n = 1$: L.H.S. $= 2 \cdot 4 = 8$, R.H.S. $= 8$. So $P(1)$ is true.

Assume $P(k)$ is true: $2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + (k + 1)(k + 3) = \frac{k(2k^2+15k+31)}{6}$.

Add the $(k + 1)$th term $(k + 2)(k + 4)$ to both sides.

L.H.S. of $P(k + 1) = \frac{k(2k^2+15k+31)}{6} + (k + 2)(k + 4)$

$= \frac{k^3}{3} + \frac{7k^2}{2} + \frac{67k}{6} + 8$

$= \frac{(k+1)(2k^2+19k+48)}{6}$, which is $\frac{n(2n^2+15n+31)}{6}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \frac{1}{9\cdot13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \frac{1}{9\cdot13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$.

For $n = 1$: L.H.S. $= \frac{1}{1\cdot5} = \frac{1}{5}$, R.H.S. $= \frac{1}{5}$. So $P(1)$ is true.

Assume $P(k)$ is true: $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \frac{1}{9\cdot13} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$.

Add the $(k + 1)$th term $\frac{1}{(4k+1)(4k+5)}$ to both sides.

L.H.S. of $P(k + 1) = \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$

$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$

$= \frac{k+1}{4k+5}$, which is $\frac{n}{4n+1}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $1^2 + 2^2 + 3^2 + \cdots + 24^2$. [1]

$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

$n = 24$: $\frac{24\cdot25\cdot49}{6} = 4900$

Ans: 4900

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6. Find $1^3 + 2^3 + 3^3 + \cdots + 18^3$. [2]

$\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

$n = 18: \ \left(\frac{18 \cdot 19}{2}\right)^2 = 171^2 = 29241$

Ans: 29241

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7. Find $1^3 + 3^3 + 5^3 + \ldots$ to 7 terms. [2]

$\sum (2k-1)^3 = n^2(2n^2 - 1)$

$n = 7: \ 49(2 \cdot 49 - 1) = 49 \cdot 97 = 4753$

Ans: 4753

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 19^2$. [1]

There are $n = 19$ terms and n is odd.

$\frac{n(n+1)}{2} = \frac{19 \cdot 20}{2} = 190$

Ans: 190

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9. Using the standard sums, find $\sum_{k=1}^{n} k(k-1)(k+1)$ in factorised form. [3]

$k(k-1)(k+1) = k^3 - k$

$\sum_{k=1}^{n}(k^3 - k) = \sum k^3 - \sum k$

$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2}$

$= \frac{n^4 + 2n^3 - n^2 - 2n}{4}$

$= \frac{n(n-1)(n+1)(n+2)}{4}$

Ans: $\frac{n(n-1)(n+1)(n+2)}{4}$

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10. Find the 22th term and the sum of the first 22 terms of $9, 13, 17, \ldots$ [2]

$a = 9$, $d = 4$.

$t_{22} = a + 21d = 9 + 21(4) = 93$

$S_{22} = \frac{22}{2}[a + l] = \frac{22}{2}(9 + 93) = 1122$

Ans: $t_{22} = 93$, $S_{22} = 1122$

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11. Find the 7th term and the sum of the first 7 terms of $1, 4, 16, \ldots$ [2]

$a = 1$, $r = 4$.

$t_7 = ar^6 = 1 \cdot 4^6 = 4096$

$S_7 = a\frac{r^7 - 1}{r - 1} = 1 \cdot \frac{16384 - 1}{3} = 5461$

Ans: $t_7 = 4096$, $S_7 = 5461$

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12. Find the sum to infinity of $4 + 5 \cdot \frac{1}{3} + 6 \cdot \frac{1}{9} + 7 \cdot \frac{1}{27} + \ldots$ [3]

This is an A.G.P. with $a = 4$, $d = 1$, $r = \frac{1}{3}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{4}{\frac{2}{3}} + \frac{\frac{1}{3}}{\frac{4}{9}}$

$= 6 + \frac{3}{4} = \frac{27}{4}$

Ans: $\frac{27}{4}$

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13. Find the nth term of the sequence $5, 7, 11, 17, \ldots$ [3]

Differences: $2, 4, 6$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 5,\ 4a + 2b + c = 7,\ 9a + 3b + c = 11$

$3a + b = 2,\ 5a + b = 4 \Rightarrow a = 1,\ b = -1,\ c = 5$

$\therefore T_n = n^2 - n + 5$ (check: $T_4 = 17$)

Ans: $T_n = n^2 - n + 5$

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14. Find the nth term of the sequence $3, 23, 123, 623, \ldots$ [3]

Differences: $20, 100, 500$, in G.P. with ratio 5, so $T_n = A \cdot 5^n + B$.

$n = 1:\ 5A + B = 3;\ n = 2:\ 25A + B = 23$

$20A = 20 \Rightarrow A = 1,\ B = -2$

$\therefore T_n = 5^n - 2$

Ans: $T_n = 5^n - 2$

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15. A series is written $1 \cdot 5 + 5 \cdot 9 + \cdots + (4n-7)(4n-3)$, that is, up to $(n-1)$ terms. With which value of n should the verification begin? [1]

The series has $(n-1)$ terms.

It has at least one term only when $(n-1) \geq 1$, that is $n \geq 2$.

So the verification begins with $n = 2$.

Ans: $n = 2$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $3 \cdot 4^n + 2^n - 2$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 12$

$n = 2:\ 50$

$\text{G.C.D.}(12, 50) = 2$

So $3 \cdot 4^n + 2^n - 2$ is divisible by 2 (induction proves it for all n).

Ans: 2

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17. Prove by mathematical induction that $10^n - 3^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $10^n - 3^n$ is divisible by 7.

For $n = 1$: the value is $7 = 7 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $10^k - 3^k = 7\lambda$ for some integer λ.

Then $10^k = 7\lambda + 3^k$.

At $n = k + 1$: $10^{k+1} - 3^{k+1} = 10(10^k) - 3 \cdot 3^k$

$= 10(7\lambda + 3^k) - 3 \cdot 3^k$

$= 70\lambda + 7 \cdot 3^k$

$= 7\left(10\lambda + 3^k\right)$, which is divisible by 7.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 7 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $9^n - 8n - 1$ is divisible by 64 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $9^n - 8n - 1$ is divisible by 64.

For $n = 1$: $9 - 8 - 1 = 0$, which is divisible by 64. So $P(1)$ is true.

Assume $P(k)$: $9^k - 8k - 1 = 64\lambda$, so $9^k = 64\lambda + 8k + 1$.

At $n = k + 1$: $9^{k+1} - 8(k+1) - 1 = 9 \cdot 9^k - 8k - 9$

$= 9(64\lambda + 8k + 1) - 8k - 9$

$= 576\lambda + 72k + 9 - 8k - 9$

$= 576\lambda + 64k = 64(9\lambda + k)$, which is divisible by 64.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 64 for all $n \in \mathbb{N}$

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19. Is $52 \equiv 7 \pmod 5$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$52 - 7 = 45$

$45 = 5 \times 9$, so yes.

Ans: Yes: $52 - 7 = 45$ is divisible by 5

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20. Using a standard result, show that $5 \cdot 6 \cdot 7$ is divisible by $3!$ and find the quotient. [2]

The product of 3 consecutive natural numbers is divisible by $3! = 6$.

$5 \cdot 6 \cdot 7 = 210 = 6 \times 35$

Ans: 35
