EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 06

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For the statement $P(n)$: $2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n-1)}{2}$, check $P(2)$. [1 mark]

2. To prove $2 + 11 + 20 + \cdots + (9n - 7) = \frac{n(9n-5)}{2}$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1 mark]

3. Prove by mathematical induction that $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + (2n)(2n + 2) = \frac{4n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $\sum_{k=1}^{60} k$. [1 mark]

6. Find $6^3 + 7^3 + \cdots + 12^3$. [2 marks]

7. Find $1^2 + 3^2 + 5^2 + \cdots + 25^2$. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 22^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} k(4k - 1)$ in factorised form. [3 marks]

10. Find the sum of the first n terms of $6 + 11 + 16 + \ldots$ [2 marks]

11. Find the sum of the first n terms of $2 + 14 + 98 + \ldots$ [2 marks]

12. Find the sum to infinity of $1 + 4 \cdot \frac{1}{2} + 7 \cdot \frac{1}{4} + 10 \cdot \frac{1}{8} + \ldots$ [3 marks]

13. Find the nth term of the sequence $2, 12, 28, 50, \ldots$ [3 marks]

14. Find the nth term of the sequence $9, 13, 21, 37, \ldots$ [3 marks]

15. In proving $3 + 6 + \cdots + 3 \cdot 2^{n-3}$ (up to $(n - 2)$ terms) is equal to a formula, at which value of n do we start? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $11^n - 5^n$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $n^3 + 3n^2 + 5n$ is divisible by 3 for all $n \in \mathbb{N}$. [5 marks]

19. Is $83 \equiv 20 \pmod{9}$? Give the reason. [1 mark]

20. 13 is odd. Using a standard result, show that $13(13^2 - 1)$ is divisible by 24 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 9, so $P(2)$ is true
2	$9k + 2$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	1830
6	5859
7	2925
8	-253
9	$\frac{n(n+1)(8n+1)}{6}$
10	$\frac{n(5n+7)}{2}$
11	$\frac{7^n}{3} - \frac{1}{3}$
12	8
13	$T_n = 3n^2 + n - 2$
14	$T_n = 2 \cdot 2^n + 5$
15	$n = 3$
16	6
17	Proved: divisible by 17 for all $n \in \mathbb{N}$
18	Proved: divisible by 3 for all $n \in \mathbb{N}$
19	Yes: $83 - 20 = 63$ is divisible by 9
20	91

Solutions

1. For the statement $P(n)$: $2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n-1)}{2}$, check $P(2)$. [1]

$P(2)$: L.H.S. $= 2 + 7 = 9$

R.H.S. at $n = 2$: $\frac{n(5n-1)}{2} = 9$

L.H.S. $=$ R.H.S., so $P(2)$ is true.

Ans: Both sides equal 9, so $P(2)$ is true

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2. To prove $2 + 11 + 20 + \cdots + (9n - 7) = \frac{n(9n-5)}{2}$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1]

The nth term is $(9n - 7)$.

Put $n = k + 1$: the $(k + 1)$th term is $9k + 2$.

Ans: $9k + 2$

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3. Prove by mathematical induction that $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + (2n)(2n + 2) = \frac{4n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + (2n)(2n + 2) = \frac{4n(n+1)(n+2)}{3}$.

For $n = 1$: L.H.S. $= 2 \cdot 4 = 8$, R.H.S. $= 8$. So $P(1)$ is true.

Assume $P(k)$ is true: $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2k(2k + 2) = \frac{4k(k+1)(k+2)}{3}$.

Add the $(k + 1)$th term $(2k + 2)(2k + 4)$ to both sides.

L.H.S. of $P(k + 1) = \frac{4k(k+1)(k+2)}{3} + (2k + 2)(2k + 4)$

$= \frac{4k^3+24k^2+44k+24}{3}$

$= \frac{4(k+1)(k+2)(k+3)}{3}$, which is $\frac{4n(n+1)(n+2)}{3}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$.

For $n = 1$: L.H.S. $= 2 = 2$, R.H.S. $= 2$. So $P(1)$ is true.

Assume $P(k)$ is true: $2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} = 3^k - 1$.

Add the $(k + 1)$th term $2 \cdot 3^k$ to both sides.

L.H.S. of $P(k + 1) = 3^k - 1 + 2 \cdot 3^k$

$= 3 \cdot 3^k - 1$

$= 3 \cdot 3^k - 1$, which is $3^n - 1$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $\sum_{k=1}^{60} k$. [1]

$\sum n = \frac{n(n+1)}{2}$

$n = 60$: $\frac{60 \cdot 61}{2} = 1830$

Ans: 1830

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6. Find $6^3 + 7^3 + \cdots + 12^3$. [2]

The sum from 1 to 12 minus the sum from 1 to 5.

$\sum_{k=1}^{12} k^3 = 6084,\ \sum_{k=1}^{5} k^3 = 225$

$\therefore \text{sum} = 6084 - 225 = 5859$

Ans: 5859

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7. Find $1^2 + 3^2 + 5^2 + \cdots + 25^2$. [2]

$2n - 1 = 25 \Rightarrow n = 13$

$\frac{n}{3}(4n^2 - 1) = \frac{13}{3}(676 - 1) = 2925$

Ans: 2925

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 22^2$. [1]

There are $n = 22$ terms and n is even.

$-\frac{n(n+1)}{2} = -\frac{22 \cdot 23}{2} = -253$

Ans: -253

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9. Using the standard sums, find $\sum_{k=1}^{n} k(4k-1)$ in factorised form. [3]

$k(4k-1) = 4k^2 - k$

$\sum_{k=1}^{n}(4k^2 - k) = 4\sum k^2 - \sum k$

$= 4 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$

$= \frac{8n^3+9n^2+n}{6}$

$= \frac{n(n+1)(8n+1)}{6}$

Ans: $\frac{n(n+1)(8n+1)}{6}$

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10. Find the sum of the first n terms of $6 + 11 + 16 + \ldots$ [2]

$a = 6$, $d = 5$.

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[12 + 5(n-1)]$

$= \frac{n(5n+7)}{2}$

Ans: $\frac{n(5n+7)}{2}$

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11. Find the sum of the first n terms of $2 + 14 + 98 + \ldots$ [2]

$a = 2$, $r = 7$.

$S_n = a\frac{r^n - 1}{r - 1} = \frac{7^n}{3} - \frac{1}{3}$

Ans: $\frac{7^n}{3} - \frac{1}{3}$

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12. Find the sum to infinity of $1 + 4 \cdot \frac{1}{2} + 7 \cdot \frac{1}{4} + 10 \cdot \frac{1}{8} + \ldots$ [3]

This is an A.G.P. with $a = 1$, $d = 3$, $r = \frac{1}{2}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{1}{\frac{1}{2}} + \frac{\frac{3}{2}}{\frac{1}{4}}$

$= 2 + 6 = 8$

Ans: 8

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13. Find the nth term of the sequence $2, 12, 28, 50, \ldots$ [3]

Differences: $10, 16, 22$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 2,\ 4a + 2b + c = 12,\ 9a + 3b + c = 28$

$3a + b = 10,\ 5a + b = 16 \Rightarrow a = 3,\ b = 1,\ c = -2$

$\therefore T_n = 3n^2 + n - 2$ (check: $T_4 = 50$)

Ans: $T_n = 3n^2 + n - 2$

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14. Find the nth term of the sequence $9, 13, 21, 37, \ldots$ [3]

Differences: $4, 8, 16$, in G.P. with ratio 2, so $T_n = A \cdot 2^n + B$.

$n = 1:\ 2A + B = 9;\ n = 2:\ 4A + B = 13$

$2A = 4 \Rightarrow A = 2,\ B = 5$

$\therefore T_n = 2 \cdot 2^n + 5$

Ans: $T_n = 2 \cdot 2^n + 5$

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15. In proving $3 + 6 + \cdots + 3 \cdot 2^{n-3}$ (up to $(n-2)$ terms) is equal to a formula, at which value of n do we start? [1]

The series has $(n-2)$ terms.

It has at least one term only when $(n-2) \geq 1$, that is $n \geq 3$.

So the verification begins with $n = 3$.

Ans: $n = 3$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $11^n - 5^n$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 6$

$n = 2:\ 96$

$\text{G.C.D.}(6, 96) = 6$

So $11^n - 5^n$ is divisible by 6 (induction proves it for all n).

Ans: 6

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17. Prove by mathematical induction that $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

For $n = 1$: the value is $391 = 17 \times 23$. So $P(1)$ is true.

Assume $P(k)$: $3 \cdot 5^{2k+1} + 2^{3k+1} = 17\lambda$ for some integer λ.

Then $3 \cdot 5^{2k+1} = 17\lambda - 2^{3k+1}$.

At $n = k + 1$: $3 \cdot 5^{2k+3} + 2^{3k+4} = 25(3 \cdot 5^{2k+1}) + 8 \cdot 2^{3k+1}$

$= 25(17\lambda - 2^{3k+1}) + 8 \cdot 2^{3k+1}$

$= 425\lambda - 17 \cdot 2^{3k+1}$

$= 17\left(25\lambda - 2^{3k+1}\right)$, which is divisible by 17.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 17 **for all** $n \in \mathbb{N}$

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18. Prove by mathematical induction that $n^3 + 3n^2 + 5n$ is divisible by 3 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $n^3 + 3n^2 + 5n$ is divisible by 3.

For $n = 1$: the value is 9, divisible by 3. So $P(1)$ is true.

Assume $P(k)$: $k^3 + 3k^2 + 5k = 3\lambda$.

At $n = k + 1$: $(k+1)^3 + 3(k+1)^2 + 5(k+1)$

$= (k^3 + 3k^2 + 5k) + (3k^2 + 9k + 9)$

$= 3\lambda + 3(k^2 + 3k + 3)$, which is divisible by 3.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 3 for all $n \in \mathbb{N}$

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19. Is $83 \equiv 20 \pmod 9$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$83 - 20 = 63$

$63 = 9 \times 7$, so yes.

Ans: Yes: $83 - 20 = 63$ is divisible by 9

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20. 13 is odd. Using a standard result, show that $13(13^2 - 1)$ is divisible by 24 and find the quotient. [2]

For odd n, $n(n^2 - 1)$ is divisible by 24.

$13(169 - 1) = 13 \times 168 = 2184 = 24 \times 91$

Ans: 91
