EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 05

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Let $P(n)$: $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$. Verify that $P(3)$ is true. [1 mark]

2. In proving $1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + \cdots + n(2n - 1) = \frac{n(n+1)(4n-1)}{6}$ by induction, what term is added to both sides of $P(k)$? [1 mark]

3. Prove by mathematical induction that $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots + (3n - 2)(3n + 1) = n(3n^2 + 3n - 2)$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{6n+9}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $1^2 + 2^2 + 3^2 + \cdots + 18^2$. [1 mark]

6. Find $1^4 + 2^4 + 3^4 + \cdots + 7^4$ using the formula. [2 marks]

7. Find $2^2 + 4^2 + 6^2 + \cdots + 30^2$. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 9^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} (k + 2)(2k - 1)$ in factorised form. [3 marks]

10. Find the sum $4 + 13 + 22 + \cdots + 130$. [2 marks]

11. Find the sum to infinity of $12 - 6 + 3 - \frac{3}{2} + \ldots$ [2 marks]

12. Using the A.G.P. formula, find $2 + 3 \cdot 3 + 4 \cdot 9 + 5 \cdot 27 + \ldots$ to 5 terms. [3 marks]

13. Find the nth term of the sequence $7, 12, 19, 28, \ldots$ [3 marks]

14. Find the nth term of the sequence $5, 17, 65, 257, \ldots$ [3 marks]

15. A series is written $3 + 13 + \cdots + (10n - 7)$, that is, up to n terms. With which value of n should the verification begin? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $2^{3n} - 1$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $10^{2n} - 1$ is divisible by 99 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $6^n - 5n - 1$ is divisible by 25 for all $n \in \mathbb{N}$. [5 marks]

19. Is $64 \equiv 9 \pmod{11}$? Give the reason. [1 mark]

20. Using a standard result, show that $7^3 + 8^3 + 9^3$ is divisible by 9 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 21, so $P(3)$ is true
2	$(k+1)(2k+1)$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	2109
6	4676
7	4960
8	45
9	$\frac{n(4n^2+15n-1)}{6}$
10	1005
11	8
12	668
13	$T_n = n^2 + 2n + 4$
14	$T_n = 4^n + 1$
15	$n = 1$
16	7
17	Proved: divisible by 99 for all $n \in \mathbb{N}$
18	Proved: divisible by 25 for all $n \in \mathbb{N}$
19	Yes: $64 - 9 = 55$ is divisible by 11
20	176

Solutions

1. Let $P(n)$: $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$. Verify that $P(3)$ is true. [1]

$P(3)$: L.H.S. $= 3 + 7 + 11 = 21$

R.H.S. at $n = 3$: $n(2n + 1) = 21$

L.H.S. $=$ R.H.S., so $P(3)$ is true.

Ans: Both sides equal 21, so $P(3)$ is true

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2. In proving $1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + \cdots + n(2n - 1) = \frac{n(n+1)(4n-1)}{6}$ by induction, what term is added to both sides of $P(k)$? [1]

The nth term is $n(2n - 1)$.

Put $n = k + 1$: the $(k + 1)$th term is $(k + 1)(2k + 1)$.

Ans: $(k + 1)(2k + 1)$

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3. Prove by mathematical induction that $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots + (3n - 2)(3n + 1) = n(3n^2 + 3n - 2)$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots + (3n - 2)(3n + 1) = n(3n^2 + 3n - 2)$.

For $n = 1$: L.H.S. $= 1 \cdot 4 = 4$, R.H.S. $= 4$. So $P(1)$ is true.

Assume $P(k)$ is true: $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots + (3k - 2)(3k + 1) = k(3k^2 + 3k - 2)$.

Add the $(k + 1)$th term $(3k + 1)(3k + 4)$ to both sides.

L.H.S. of $P(k + 1) = k(3k^2 + 3k - 2) + (3k + 1)(3k + 4)$

$= 3k^3 + 12k^2 + 13k + 4$

$= (k + 1)(3k^2 + 9k + 4)$, which is $n(3n^2 + 3n - 2)$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{6n+9}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{6n+9}$.

For $n = 1$: L.H.S. $= \frac{1}{3 \cdot 5} = \frac{1}{15}$, R.H.S. $= \frac{1}{15}$. So $P(1)$ is true.

Assume $P(k)$ is true: $\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{6k+9}$.

Add the $(k + 1)$th term $\frac{1}{(2k+3)(2k+5)}$ to both sides.

L.H.S. of $P(k + 1) = \frac{k}{6k+9} + \frac{1}{(2k+3)(2k+5)}$

$= \frac{2k^2+5k+3}{3(2k+3)(2k+5)}$

$= \frac{k+1}{3(2k+5)}$, which is $\frac{n}{6n+9}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $1^2 + 2^2 + 3^2 + \cdots + 18^2$. [1]

$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

$n = 18$: $\frac{18 \cdot 19 \cdot 37}{6} = 2109$

Ans: 2109

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6. Find $1^4 + 2^4 + 3^4 + \cdots + 7^4$ using the formula. [2]

$\sum n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

$n = 7: \ \frac{7 \cdot 8 \cdot 15 \cdot 167}{30}$

$= 4676$

Ans: 4676

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7. Find $2^2 + 4^2 + 6^2 + \cdots + 30^2$. [2]

The last term is $(2n)^2$ with $n = 15$.

$\sum (2k)^2 = \frac{2}{3}n(n+1)(2n+1) = \frac{2}{3} \cdot 15 \cdot 16 \cdot 31 = 4960$

Ans: 4960

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 9^2$. [1]

There are $n = 9$ terms and n is odd.

$\frac{n(n+1)}{2} = \frac{9 \cdot 10}{2} = 45$

Ans: 45

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9. Using the standard sums, find $\sum_{k=1}^{n}(k+2)(2k-1)$ in factorised form. [3]

$(k+2)(2k-1) = 2k^2 + 3k - 2$

$\sum_{k=1}^{n}(2k^2 + 3k - 2) = 2\sum k^2 + 3\sum k - 2\sum 1$

$= 2 \cdot \frac{n(n+1)(2n+1)}{6} + 3 \cdot \frac{n(n+1)}{2} - 2 \cdot n$

$= \frac{4n^3 + 15n^2 - n}{6}$

$= \frac{n(4n^2 + 15n - 1)}{6}$

Ans: $\frac{n(4n^2 + 15n - 1)}{6}$

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10. Find the sum $4 + 13 + 22 + \cdots + 130$. [2]

$a = 4$, $d = 9$, $l = 130$.

$130 = 4 + (n-1)9 \Rightarrow n = 15$

$S_n = \frac{n}{2}[a + l] = \frac{15}{2}(4 + 130) = 1005$

Ans: 1005

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11. Find the sum to infinity of $12 - 6 + 3 - \frac{3}{2} + \ldots$ [2]

$a = 12$, $r = -\frac{1}{2}$, $|r| < 1$.

$S_\infty = \frac{a}{1-r} = \frac{12}{1-(-\frac{1}{2})} = 8$

Ans: 8

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12. Using the A.G.P. formula, find $2 + 3 \cdot 3 + 4 \cdot 9 + 5 \cdot 27 + \ldots$ to 5 terms. [3]

$a = 2$, $d = 1$, $r = 3$, $n = 5$; the last term has $a + (n-1)d = 6$.

$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$

$= \frac{2}{-2} + \frac{3(1-81)}{4} - \frac{6 \cdot 243}{-2}$

$= 668$

Ans: 668

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13. Find the nth term of the sequence $7, 12, 19, 28, \ldots$ [3]

Differences: $5, 7, 9$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 7,\ 4a + 2b + c = 12,\ 9a + 3b + c = 19$

$3a + b = 5,\ 5a + b = 7 \Rightarrow a = 1,\ b = 2,\ c = 4$

$\therefore T_n = n^2 + 2n + 4$ (check: $T_4 = 28$)

Ans: $T_n = n^2 + 2n + 4$

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14. Find the nth term of the sequence $5, 17, 65, 257, \ldots$ [3]

Differences: $12, 48, 192$, in G.P. with ratio 4, so $T_n = A \cdot 4^n + B$.

$n = 1:\ 4A + B = 5;\ n = 2:\ 16A + B = 17$

$12A = 12 \Rightarrow A = 1,\ B = 1$

$\therefore T_n = 4^n + 1$

Ans: $T_n = 4^n + 1$

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15. A series is written $3 + 13 + \cdots + (10n - 7)$, that is, up to n terms. With which value of n should the verification begin? [1]

The series has n terms.

It has at least one term only when $n \geq 1$, that is $n \geq 1$.

So the verification begins with $n = 1$.

Ans: $n = 1$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $2^{3n} - 1$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 7$

$n = 2:\ 63$

$\text{G.C.D.}(7, 63) = 7$

So $2^{3n} - 1$ is divisible by 7 (induction proves it for all n).

Ans: 7

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17. Prove by mathematical induction that $10^{2n} - 1$ is divisible by 99 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $10^{2n} - 1$ is divisible by 99.

For $n = 1$: the value is $99 = 99 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $10^{2k} - 1 = 99\lambda$ for some integer λ.

Then $10^{2k} = 99\lambda + 1$.

At $n = k + 1$: $10^{2k+2} - 1 = 100(10^{2k}) - 1$

$= 100(99\lambda + 1) - 1$

$= 9900\lambda + 99$

$= 99\,(100\lambda + 1)$, which is divisible by 99.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 99 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $6^n - 5n - 1$ is divisible by 25 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $6^n - 5n - 1$ is divisible by 25.

For $n = 1$: $6 - 5 - 1 = 0$, which is divisible by 25. So $P(1)$ is true.

Assume $P(k)$: $6^k - 5k - 1 = 25\lambda$, so $6^k = 25\lambda + 5k + 1$.

At $n = k + 1$: $6^{k+1} - 5(k + 1) - 1 = 6 \cdot 6^k - 5k - 6$

$= 6(25\lambda + 5k + 1) - 5k - 6$

$= 150\lambda + 30k + 6 - 5k - 6$

$= 150\lambda + 25k = 25(6\lambda + k)$, which is divisible by 25.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 25 for all $n \in \mathbb{N}$

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19. Is $64 \equiv 9 \pmod{11}$? Give the reason. [1]

$x \equiv y \pmod{m}$ means $x - y$ is divisible by m.

$64 - 9 = 55$

$55 = 11 \times 5$, so yes.

Ans: Yes: $64 - 9 = 55$ is divisible by 11

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20. Using a standard result, show that $7^3 + 8^3 + 9^3$ is divisible by 9 and find the quotient. [2]

The sum of the cubes of three consecutive natural numbers is divisible by 9.

$343 + 512 + 729 = 1584 = 9 \times 176$

Ans: 176
