EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For the statement $P(n)$: $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$, check $P(4)$. [1 mark]

2. To prove $5 + 10 + 20 + \cdots + 5 \cdot 2^{n-1} = 5(2^n - 1)$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1 mark]

3. Prove by mathematical induction that $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n - 1)(2n + 1) = \frac{n(4n^2+6n-1)}{3}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{5^n-1}{4}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $\sum_{k=1}^{48} k$. [1 mark]

6. Find $1^3 + 2^3 + 3^3 + \cdots + 15^3$. [2 marks]

7. Find $1 + 3 + 5 + \cdots + 59$. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 16^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} k^2(k + 3)$ in factorised form. [3 marks]

10. Find the 30th term and the sum of the first 30 terms of $11, 16, 21, \ldots$ [2 marks]

11. Find the 6th term and the sum of the first 6 terms of $4, 12, 36, \ldots$ [2 marks]

12. Find the sum to infinity of $1 + 5 \cdot \frac{1}{5} + 9 \cdot \frac{1}{25} + 13 \cdot \frac{1}{125} + \ldots$ [3 marks]

13. Find the nth term of the sequence $3, 6, 13, 24, \ldots$ [3 marks]

14. Find the nth term of the sequence $5, 11, 23, 47, \ldots$ [3 marks]

15. In proving $2 \cdot 5 + 5 \cdot 8 + \cdots + (3n - 10)(3n - 7)$ (up to $(n - 3)$ terms) is equal to a formula, at which value of n do we start? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $7^n - 1$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \in \mathbb{N}$. [5 marks]

19. Is $71 \equiv 26 \pmod{8}$? Give the reason. [1 mark]

20. Using Fermat's result for the prime 5, show that $3^5 - 3$ is divisible by 5 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 24, so $P(4)$ is true
2	$5 \cdot 2^k$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	1176
6	14400
7	900
8	-136
9	$\frac{n(n+1)(n^2+5n+2)}{4}$
10	$t_{30} = 156$, $S_{30} = 2505$
11	$t_6 = 972$, $S_6 = 1456$
12	$\frac{5}{2}$
13	$T_n = 2n^2 - 3n + 4$
14	$T_n = 3 \cdot 2^n - 1$
15	$n = 4$
16	6
17	Proved: divisible by 7 for all $n \in \mathbb{N}$
18	Proved: divisible by 6 for all $n \in \mathbb{N}$
19	No: $71 - 26 = 45$ is not divisible by 8
20	48

Solutions

1. For the statement $P(n)$: $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$, check $P(4)$. [1]

$P(4)$: L.H.S. $= 3 + 5 + 7 + 9 = 24$

R.H.S. at $n = 4$: $n(n + 2) = 24$

L.H.S. $=$ R.H.S., so $P(4)$ is true.

Ans: Both sides equal 24, so $P(4)$ is true

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2. To prove $5 + 10 + 20 + \cdots + 5 \cdot 2^{n-1} = 5(2^n - 1)$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1]

The nth term is $5 \cdot 2^{n-1}$.

Put $n = k + 1$: the $(k + 1)$th term is $5 \cdot 2^k$.

Ans: $5 \cdot 2^k$

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3. Prove by mathematical induction that $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n - 1)(2n + 1) = \frac{n(4n^2+6n-1)}{3}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n - 1)(2n + 1) = \frac{n(4n^2+6n-1)}{3}$.

For $n = 1$: L.H.S. $= 1 \cdot 3 = 3$, R.H.S. $= 3$. So $P(1)$ is true.

Assume $P(k)$ is true: $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2k - 1)(2k + 1) = \frac{k(4k^2+6k-1)}{3}$.

Add the $(k + 1)$th term $(2k + 1)(2k + 3)$ to both sides.

L.H.S. of $P(k + 1) = \frac{k(4k^2+6k-1)}{3} + (2k + 1)(2k + 3)$

$= \frac{4k^3}{3} + 6k^2 + \frac{23k}{3} + 3$

$= \frac{(k+1)(4k^2+14k+9)}{3}$, which is $\frac{n(4n^2+6n-1)}{3}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{5^n-1}{4}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{5^n-1}{4}$.

For $n = 1$: L.H.S. $= 1 = 1$, R.H.S. $= 1$. So $P(1)$ is true.

Assume $P(k)$ is true: $1 + 5 + 25 + \cdots + 5^{k-1} = \frac{5^k}{4} - \frac{1}{4}$.

Add the $(k + 1)$th term 5^k to both sides.

L.H.S. of $P(k + 1) = \frac{5^k}{4} - \frac{1}{4} + 5^k$

$= \frac{5 \cdot 5^k}{4} - \frac{1}{4}$

$= \frac{5 \cdot 5^k - 1}{4}$, which is $\frac{5^n-1}{4}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $\sum_{k=1}^{48} k$. [1]

$\sum n = \frac{n(n+1)}{2}$

$n = 48$: $\frac{48 \cdot 49}{2} = 1176$

Ans: 1176

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6. Find $1^3 + 2^3 + 3^3 + \cdots + 15^3$. [2]

$\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

$n = 15: \ \left(\frac{15 \cdot 16}{2}\right)^2 = 120^2 = 14400$

Ans: 14400

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7. Find $1 + 3 + 5 + \cdots + 59$. [2]

$2n - 1 = 59 \Rightarrow n = 30$

$1 + 3 + \cdots + (2n-1) = n^2 = 30^2 = 900$

Ans: 900

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 16^2$. [1]

There are $n = 16$ terms and n is even.

$-\frac{n(n+1)}{2} = -\frac{16 \cdot 17}{2} = -136$

Ans: -136

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9. Using the standard sums, find $\sum_{k=1}^{n} k^2(k+3)$ in factorised form. [3]

$k^2(k+3) = k^3 + 3k^2$

$\sum_{k=1}^{n}(k^3 + 3k^2) = \sum k^3 + 3\sum k^2$

$= \frac{n^2(n+1)^2}{4} + 3 \cdot \frac{n(n+1)(2n+1)}{6}$

$= \frac{n^4 + 6n^3 + 7n^2 + 2n}{4}$

$= \frac{n(n+1)(n^2+5n+2)}{4}$

Ans: $\frac{n(n+1)(n^2+5n+2)}{4}$

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10. Find the 30th term and the sum of the first 30 terms of $11, 16, 21, \ldots$ [2]

$a = 11$, $d = 5$.

$t_{30} = a + 29d = 11 + 29(5) = 156$

$S_{30} = \frac{30}{2}[a + l] = \frac{30}{2}(11 + 156) = 2505$

Ans: $t_{30} = 156$, $S_{30} = 2505$

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11. Find the 6th term and the sum of the first 6 terms of $4, 12, 36, \ldots$ [2]

$a = 4$, $r = 3$.

$t_6 = ar^5 = 4 \cdot 3^5 = 972$

$S_6 = a\frac{r^6 - 1}{r - 1} = 4 \cdot \frac{729 - 1}{2} = 1456$

Ans: $t_6 = 972$, $S_6 = 1456$

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12. Find the sum to infinity of $1 + 5 \cdot \frac{1}{5} + 9 \cdot \frac{1}{25} + 13 \cdot \frac{1}{125} + \ldots$ [3]

This is an A.G.P. with $a = 1$, $d = 4$, $r = \frac{1}{5}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{1}{\frac{4}{5}} + \frac{\frac{4}{5}}{\frac{16}{25}}$

$= \frac{5}{4} + \frac{5}{4} = \frac{5}{2}$

Ans: $\frac{5}{2}$

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13. Find the nth term of the sequence $3, 6, 13, 24, \ldots$ [3]

Differences: $3, 7, 11$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 3,\ 4a + 2b + c = 6,\ 9a + 3b + c = 13$

$3a + b = 3,\ 5a + b = 7 \Rightarrow a = 2,\ b = -3,\ c = 4$

$\therefore T_n = 2n^2 - 3n + 4$ (check: $T_4 = 24$)

Ans: $T_n = 2n^2 - 3n + 4$

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14. Find the nth term of the sequence $5, 11, 23, 47, \ldots$ [3]

Differences: $6, 12, 24$, in G.P. with ratio 2, so $T_n = A \cdot 2^n + B$.

$n = 1:\ 2A + B = 5;\ n = 2:\ 4A + B = 11$

$2A = 6 \Rightarrow A = 3,\ B = -1$

$\therefore T_n = 3 \cdot 2^n - 1$

Ans: $T_n = 3 \cdot 2^n - 1$

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15. In proving $2 \cdot 5 + 5 \cdot 8 + \cdots + (3n - 10)(3n - 7)$ (up to $(n - 3)$ terms) is equal to a formula, at which value of n do we start? [1]

The series has $(n - 3)$ terms.

It has at least one term only when $(n - 3) \geq 1$, that is $n \geq 4$.

So the verification begins with $n = 4$.

Ans: $n = 4$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $7^n - 1$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 6$

$n = 2:\ 48$

$\text{G.C.D.}(6, 48) = 6$

So $7^n - 1$ is divisible by 6 (induction proves it for all n).

Ans: 6

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17. Prove by mathematical induction that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $11^n - 4^n$ is divisible by 7.

For $n = 1$: the value is $7 = 7 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $11^k - 4^k = 7\lambda$ for some integer λ.

Then $11^k = 7\lambda + 4^k$.

At $n = k + 1$: $11^{k+1} - 4^{k+1} = 11(11^k) - 4 \cdot 4^k$

$= 11(7\lambda + 4^k) - 4 \cdot 4^k$

$= 77\lambda + 7 \cdot 4^k$

$= 7\left(11\lambda + 4^k\right)$, which is divisible by 7.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 7 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2n^3 + 3n^2 + n$ is divisible by 6.

For $n = 1$: the value is 6, divisible by 6. So $P(1)$ is true.

Assume $P(k)$: $2k^3 + 3k^2 + k = 6\lambda$.

At $n = k + 1$: $2(k+1)^3 + 3(k+1)^2 + (k+1)$

$= (2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)$

$= 6\lambda + 6(k^2 + 2k + 1)$, which is divisible by 6.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 6 for all $n \in \mathbb{N}$

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19. Is $71 \equiv 26 \pmod 8$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$71 - 26 = 45$

45 leaves remainder 5 on division by 8, so no.

Ans: No: $71 - 26 = 45$ is not divisible by 8

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20. Using Fermat's result for the prime 5, show that $3^5 - 3$ is divisible by 5 and find the quotient. [2]

For a prime p, $n^p - n$ is divisible by p.

$3^5 - 3 = 243 - 3 = 240 = 5 \times 48$

Ans: 48
