EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Let $P(n)$: $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$. Verify that $P(2)$ is true. [1 mark]

2. In proving $1^2 + 5^2 + 9^2 + \cdots + (4n - 3)^2 = \frac{n(16n^2-12n-1)}{3}$ by induction, what term is added to both sides of $P(k)$? [1 mark]

3. Prove by mathematical induction that $2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (n + 1)(n + 2) = \frac{n(n^2+6n+11)}{3}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $1^2 + 2^2 + 3^2 + \cdots + 15^2$. [1 mark]

6. Find $11^2 + 12^2 + \cdots + 25^2$. [2 marks]

7. Find $1^3 + 3^3 + 5^3 + \ldots$ to 6 terms. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 15^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} k(k + 1)^2$ in factorised form. [3 marks]

10. Find the sum of the first n terms of $2 + 9 + 16 + \ldots$ [2 marks]

11. Find the sum of the first n terms of $5 + 15 + 45 + \ldots$ [2 marks]

12. Find the sum to infinity of $3 + 5 \cdot \frac{1}{2} + 7 \cdot \frac{1}{4} + 9 \cdot \frac{1}{8} + \ldots$ [3 marks]

13. Find the nth term of the sequence $4, 12, 26, 46, \ldots$ [3 marks]

14. Find the nth term of the sequence $5, 11, 29, 83, \ldots$ [3 marks]

15. A series is written $2^2 + 5^2 + \cdots + (3n - 4)^2$, that is, up to $(n - 1)$ terms. With which value of n should the verification begin? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $8^n - 3^n$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $3^{2n-1} + 5^{2n-1}$ is divisible by 8 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$. [5 marks]

19. Is $59 \equiv 14 \pmod{9}$? Give the reason. [1 mark]

20. Using a standard result, show that $6 \cdot 7 \cdot 8 \cdot 9$ is divisible by $4!$ and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 5, so $P(2)$ is true
2	$(4k+1)^2$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	1240
6	5140
7	2556
8	120
9	$\frac{n(n+1)(n+2)(3n+5)}{12}$
10	$\frac{n(7n-3)}{2}$
11	$\frac{5\cdot 3^n}{2} - \frac{5}{2}$
12	10
13	$T_n = 3n^2 - n + 2$
14	$T_n = 3^n + 2$
15	$n = 2$
16	5
17	Proved: divisible by 8 for all $n \in \mathbb{N}$
18	Proved: divisible by 16 for all $n \in \mathbb{N}$
19	Yes: $59 - 14 = 45$ is divisible by 9
20	126

Solutions

1. Let $P(n)$: $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$. Verify that $P(2)$ is true. [1]

$P(2)$: L.H.S. $= 1 + 4 = 5$
R.H.S. at $n = 2$: $\frac{n(3n-1)}{2} = 5$
L.H.S. $=$ R.H.S., so $P(2)$ is true.

Ans: Both sides equal 5, so $P(2)$ is true

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2. In proving $1^2 + 5^2 + 9^2 + \cdots + (4n - 3)^2 = \frac{n(16n^2-12n-1)}{3}$ by induction, what term is added to both sides of $P(k)$? [1]

The nth term is $(4n - 3)^2$.
Put $n = k + 1$: the $(k + 1)$th term is $(4k + 1)^2$.

Ans: $(4k + 1)^2$

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3. Prove by mathematical induction that $2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (n + 1)(n + 2) = \frac{n(n^2+6n+11)}{3}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (n + 1)(n + 2) = \frac{n(n^2+6n+11)}{3}$.
For $n = 1$: L.H.S. $= 2 \cdot 3 = 6$, R.H.S. $= 6$. So $P(1)$ is true.
Assume $P(k)$ is true: $2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (k + 1)(k + 2) = \frac{k(k^2+6k+11)}{3}$.
Add the $(k + 1)$th term $(k + 2)(k + 3)$ to both sides.
L.H.S. of $P(k + 1) = \frac{k(k^2+6k+11)}{3} + (k + 2)(k + 3)$
$= \frac{k^3}{3} + 3k^2 + \frac{26k}{3} + 6$
$= \frac{(k+1)(k^2+8k+18)}{3}$, which is $\frac{n(n^2+6n+11)}{3}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.
So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$.
For $n = 1$: L.H.S. $= \frac{1}{1\cdot4} = \frac{1}{4}$, R.H.S. $= \frac{1}{4}$. So $P(1)$ is true.
Assume $P(k)$ is true: $\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \cdots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$.
Add the $(k + 1)$th term $\frac{1}{(3k+1)(3k+4)}$ to both sides.
L.H.S. of $P(k + 1) = \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$
$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$
$= \frac{k+1}{3k+4}$, which is $\frac{n}{3n+1}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.
So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $1^2 + 2^2 + 3^2 + \cdots + 15^2$. [1]

$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
$n = 15$: $\frac{15\cdot16\cdot31}{6} = 1240$

Ans: 1240

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6. Find $11^2 + 12^2 + \cdots + 25^2$. [2]

The sum from 1 to 25 minus the sum from 1 to 10.

$\sum_{k=1}^{25} k^2 = 5525,\ \sum_{k=1}^{10} k^2 = 385$

$\therefore \text{sum} = 5525 - 385 = 5140$

Ans: 5140

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7. Find $1^3 + 3^3 + 5^3 + \ldots$ to 6 terms. [2]

$\sum(2k-1)^3 = n^2(2n^2-1)$

$n = 6:\ 36(2 \cdot 36 - 1) = 36 \cdot 71 = 2556$

Ans: 2556

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 15^2$. [1]

There are $n = 15$ terms and n is odd.

$\frac{n(n+1)}{2} = \frac{15 \cdot 16}{2} = 120$

Ans: 120

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9. Using the standard sums, find $\sum_{k=1}^{n} k(k+1)^2$ in factorised form. [3]

$k(k+1)^2 = k^3 + 2k^2 + k$

$\sum_{k=1}^{n}(k^3 + 2k^2 + k) = \sum k^3 + 2\sum k^2 + \sum k$

$= \frac{n^2(n+1)^2}{4} + 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{3n^4+14n^3+21n^2+10n}{12}$

$= \frac{n(n+1)(n+2)(3n+5)}{12}$

Ans: $\frac{n(n+1)(n+2)(3n+5)}{12}$

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10. Find the sum of the first n terms of $2 + 9 + 16 + \ldots$ [2]

$a = 2$, $d = 7$.

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[4 + 7(n-1)]$

$= \frac{n(7n-3)}{2}$

Ans: $\frac{n(7n-3)}{2}$

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11. Find the sum of the first n terms of $5 + 15 + 45 + \ldots$ [2]

$a = 5$, $r = 3$.

$S_n = a\frac{r^n - 1}{r-1} = \frac{5 \cdot 3^n}{2} - \frac{5}{2}$

Ans: $\frac{5 \cdot 3^n}{2} - \frac{5}{2}$

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12. Find the sum to infinity of $3 + 5 \cdot \frac{1}{2} + 7 \cdot \frac{1}{4} + 9 \cdot \frac{1}{8} + \ldots$ [3]

This is an A.G.P. with $a = 3$, $d = 2$, $r = \frac{1}{2}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{3}{\frac{1}{2}} + \frac{1}{\frac{1}{4}}$

$= 6 + 4 = 10$

Ans: 10

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13. Find the nth term of the sequence $4, 12, 26, 46, \ldots$ [3]

Differences: $8, 14, 20$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 4,\ 4a + 2b + c = 12,\ 9a + 3b + c = 26$

$3a + b = 8,\ 5a + b = 14 \Rightarrow a = 3,\ b = -1,\ c = 2$

$\therefore T_n = 3n^2 - n + 2$ (check: $T_4 = 46$)

Ans: $T_n = 3n^2 - n + 2$

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14. Find the nth term of the sequence $5, 11, 29, 83, \ldots$ [3]

Differences: $6, 18, 54$, in G.P. with ratio 3, so $T_n = A \cdot 3^n + B$.

$n = 1:\ 3A + B = 5;\ n = 2:\ 9A + B = 11$

$6A = 6 \Rightarrow A = 1,\ B = 2$

$\therefore T_n = 3^n + 2$

Ans: $T_n = 3^n + 2$

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15. A series is written $2^2 + 5^2 + \cdots + (3n-4)^2$, that is, up to $(n-1)$ terms. With which value of n should the verification begin? [1]

The series has $(n-1)$ terms.

It has at least one term only when $(n-1) \geq 1$, that is $n \geq 2$.

So the verification begins with $n = 2$.

Ans: $n = 2$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $8^n - 3^n$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 5$

$n = 2:\ 55$

$\text{G.C.D.}(5, 55) = 5$

So $8^n - 3^n$ is divisible by 5 (induction proves it for all n).

Ans: 5

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17. Prove by mathematical induction that $3^{2n-1} + 5^{2n-1}$ is divisible by 8 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $3^{2n-1} + 5^{2n-1}$ is divisible by 8.

For $n = 1$: the value is $8 = 8 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $3^{2k-1} + 5^{2k-1} = 8\lambda$ for some integer λ.

Then $3^{2k-1} = 8\lambda - 5^{2k-1}$.

At $n = k+1$: $3^{2k+1} + 5^{2k+1} = 9(3^{2k-1}) + 25 \cdot 5^{2k-1}$

$= 9(8\lambda - 5^{2k-1}) + 25 \cdot 5^{2k-1}$

$= 72\lambda + 16 \cdot 5^{2k-1}$

$= 8\left(9\lambda + 2 \cdot 5^{2k-1}\right)$, which is divisible by 8.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 8 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $5^n - 4n - 1$ is divisible by 16.

For $n = 1$: $5 - 4 - 1 = 0$, which is divisible by 16. So $P(1)$ is true.

Assume $P(k)$: $5^k - 4k - 1 = 16\lambda$, so $5^k = 16\lambda + 4k + 1$.

At $n = k + 1$: $5^{k+1} - 4(k+1) - 1 = 5 \cdot 5^k - 4k - 5$

$= 5(16\lambda + 4k + 1) - 4k - 5$

$= 80\lambda + 20k + 5 - 4k - 5$

$= 80\lambda + 16k = 16(5\lambda + k)$, which is divisible by 16.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 16 for all $n \in \mathbb{N}$

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19. Is $59 \equiv 14 \pmod 9$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$59 - 14 = 45$

$45 = 9 \times 5$, so yes.

Ans: Yes: $59 - 14 = 45$ is divisible by 9

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20. Using a standard result, show that $6 \cdot 7 \cdot 8 \cdot 9$ is divisible by $4!$ and find the quotient. [2]

The product of 4 consecutive natural numbers is divisible by $4! = 24$.

$6 \cdot 7 \cdot 8 \cdot 9 = 3024 = 24 \times 126$

Ans: 126
