EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For the statement $P(n)$: $3 + 8 + 13 + \cdots + (5n - 2) = \frac{n(5n+1)}{2}$, check $P(4)$. [1 mark]

2. To prove $2 + 8 + 14 + \cdots + (6n - 4) = n(3n - 1)$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1 mark]

3. Prove by mathematical induction that $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n + 3) = \frac{n(n+1)(n+5)}{3}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $3 + 9 + 27 + \cdots + 3 \cdot 3^{n-1} = \frac{3(3^n - 1)}{2}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $\sum_{k=1}^{35} k$. [1 mark]

6. Find $1^4 + 2^4 + 3^4 + \cdots + 6^4$ using the formula. [2 marks]

7. Find $2^2 + 4^2 + 6^2 + \cdots + 18^2$. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 12^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} (k + 1)(2k + 1)$ in factorised form. [3 marks]

10. Find the sum $7 + 10 + 13 + \cdots + 58$. [2 marks]

11. Find the sum to infinity of $27 + 9 + 3 + 1 + \ldots$ [2 marks]

12. Using the A.G.P. formula, find $1 + 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 + \ldots$ to 6 terms. [3 marks]

13. Find the nth term of the sequence $3, 9, 17, 27, \ldots$ [3 marks]

14. Find the nth term of the sequence $5, 17, 53, 161, \ldots$ [3 marks]

15. In proving $7 + 9 + \cdots + (2n + 1)$ (up to $(n - 2)$ terms) is equal to a formula, at which value of n do we start? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $6^{2n} - 1$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$. [5 marks]

19. Is $38 \equiv 5 \pmod{7}$? Give the reason. [1 mark]

20. 9 is odd. Using a standard result, show that $9(9^2 - 1)$ is divisible by 24 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 42, so $P(4)$ is true
2	$6k + 2$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	630
6	2275
7	1140
8	-78
9	$\frac{n(4n^2+15n+17)}{6}$
10	585
11	$\frac{81}{2}$
12	321
13	$T_n = n^2 + 3n - 1$
14	$T_n = 2 \cdot 3^n - 1$
15	$n = 3$
16	35
17	Proved: divisible by 24 for all $n \in \mathbb{N}$
18	Proved: divisible by 3 for all $n \in \mathbb{N}$
19	No: $38 - 5 = 33$ is not divisible by 7
20	30

Solutions

1. For the statement $P(n)$: $3 + 8 + 13 + \cdots + (5n - 2) = \frac{n(5n+1)}{2}$, check $P(4)$. [1]

$P(4)$: L.H.S. $= 3 + 8 + 13 + 18 = 42$
R.H.S. at $n = 4$: $\frac{n(5n+1)}{2} = 42$
L.H.S. $=$ R.H.S., so $P(4)$ is true.

Ans: Both sides equal 42, so $P(4)$ is true

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2. To prove $2 + 8 + 14 + \cdots + (6n - 4) = n(3n - 1)$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1]

The nth term is $(6n - 4)$.
Put $n = k + 1$: the $(k + 1)$th term is $6k + 2$.

Ans: $6k + 2$

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3. Prove by mathematical induction that $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n + 3) = \frac{n(n+1)(n+5)}{3}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n + 3) = \frac{n(n+1)(n+5)}{3}$.
For $n = 1$: L.H.S. $= 1 \cdot 4 = 4$, R.H.S. $= 4$. So $P(1)$ is true.
Assume $P(k)$ is true: $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + k(k + 3) = \frac{k(k+1)(k+5)}{3}$.
Add the $(k + 1)$th term $(k + 1)(k + 4)$ to both sides.
L.H.S. of $P(k + 1) = \frac{k(k+1)(k+5)}{3} + (k + 1)(k + 4)$
$= \frac{k^3+9k^2+20k+12}{3}$
$= \frac{(k+1)(k+2)(k+6)}{3}$, which is $\frac{n(n+1)(n+5)}{3}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.
So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $3 + 9 + 27 + \cdots + 3 \cdot 3^{n-1} = \frac{3(3^n-1)}{2}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $3 + 9 + 27 + \cdots + 3 \cdot 3^{n-1} = \frac{3(3^n-1)}{2}$.
For $n = 1$: L.H.S. $= 3 = 3$, R.H.S. $= 3$. So $P(1)$ is true.
Assume $P(k)$ is true: $3 + 9 + 27 + \cdots + 3 \cdot 3^{k-1} = \frac{3 \cdot 3^k}{2} - \frac{3}{2}$.
Add the $(k + 1)$th term $3 \cdot 3^k$ to both sides.
L.H.S. of $P(k + 1) = \frac{3 \cdot 3^k}{2} - \frac{3}{2} + 3 \cdot 3^k$
$= \frac{9 \cdot 3^k}{2} - \frac{3}{2}$
$= \frac{3(3 \cdot 3^k-1)}{2}$, which is $\frac{3(3^n-1)}{2}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.
So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $\sum_{k=1}^{35} k$. [1]

$\sum n = \frac{n(n+1)}{2}$
$n = 35$: $\frac{35 \cdot 36}{2} = 630$

Ans: 630

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6. Find $1^4 + 2^4 + 3^4 + \cdots + 6^4$ using the formula. [2]

$\sum n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

$n = 6: \ \frac{6 \cdot 7 \cdot 13 \cdot 125}{30}$

$= 2275$

Ans: 2275

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7. Find $2^2 + 4^2 + 6^2 + \cdots + 18^2$. [2]

The last term is $(2n)^2$ with $n = 9$.

$\sum (2k)^2 = \frac{2}{3}n(n+1)(2n+1) = \frac{2}{3} \cdot 9 \cdot 10 \cdot 19 = 1140$

Ans: 1140

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 12^2$. [1]

There are $n = 12$ terms and n is even.

$-\frac{n(n+1)}{2} = -\frac{12 \cdot 13}{2} = -78$

Ans: -78

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9. Using the standard sums, find $\sum_{k=1}^{n}(k+1)(2k+1)$ in factorised form. [3]

$(k+1)(2k+1) = 2k^2 + 3k + 1$

$\sum_{k=1}^{n}(2k^2 + 3k + 1) = 2\sum k^2 + 3\sum k + \sum 1$

$= 2 \cdot \frac{n(n+1)(2n+1)}{6} + 3 \cdot \frac{n(n+1)}{2} + n$

$= \frac{4n^3 + 15n^2 + 17n}{6}$

$= \frac{n(4n^2 + 15n + 17)}{6}$

Ans: $\frac{n(4n^2 + 15n + 17)}{6}$

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10. Find the sum $7 + 10 + 13 + \cdots + 58$. [2]

$a = 7$, $d = 3$, $l = 58$.

$58 = 7 + (n-1)3 \Rightarrow n = 18$

$S_n = \frac{n}{2}[a + l] = \frac{18}{2}(7 + 58) = 585$

Ans: 585

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11. Find the sum to infinity of $27 + 9 + 3 + 1 + \ldots$ [2]

$a = 27$, $r = \frac{1}{3}$, $|r| < 1$.

$S_\infty = \frac{a}{1-r} = \frac{27}{1-(\frac{1}{3})} = \frac{81}{2}$

Ans: $\frac{81}{2}$

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12. Using the A.G.P. formula, find $1 + 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 + \ldots$ to 6 terms. [3]

$a = 1$, $d = 1$, $r = 2$, $n = 6$; the last term has $a + (n-1)d = 6$.

$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$

$= \frac{1}{-1} + \frac{2(1-32)}{1} - \frac{6 \cdot 64}{-1}$

$= 321$

Ans: 321

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13. Find the nth term of the sequence $3, 9, 17, 27, \ldots$ [3]

Differences: $6, 8, 10$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 3,\ 4a + 2b + c = 9,\ 9a + 3b + c = 17$

$3a + b = 6,\ 5a + b = 8 \Rightarrow a = 1,\ b = 3,\ c = -1$

$\therefore T_n = n^2 + 3n - 1$ (check: $T_4 = 27$)

Ans: $T_n = n^2 + 3n - 1$

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14. Find the nth term of the sequence $5, 17, 53, 161, \ldots$ [3]

Differences: $12, 36, 108$, in G.P. with ratio 3, so $T_n = A \cdot 3^n + B$.

$n = 1:\ 3A + B = 5;\ n = 2:\ 9A + B = 17$

$6A = 12 \Rightarrow A = 2,\ B = -1$

$\therefore T_n = 2 \cdot 3^n - 1$

Ans: $T_n = 2 \cdot 3^n - 1$

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15. In proving $7 + 9 + \cdots + (2n + 1)$ (up to $(n - 2)$ terms) is equal to a formula, at which value of n do we start? [1]

The series has $(n - 2)$ terms.

It has at least one term only when $(n - 2) \geq 1$, that is $n \geq 3$.

So the verification begins with $n = 3$.

Ans: $n = 3$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $6^{2n} - 1$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 35$

$n = 2:\ 1295$

$\text{G.C.D.}(35, 1295) = 35$

So $6^{2n} - 1$ is divisible by 35 (induction proves it for all n).

Ans: 35

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17. Prove by mathematical induction that $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $5^{2n} - 1$ is divisible by 24.

For $n = 1$: the value is $24 = 24 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $5^{2k} - 1 = 24\lambda$ for some integer λ.

Then $5^{2k} = 24\lambda + 1$.

At $n = k + 1$: $5^{2k+2} - 1 = 25(5^{2k}) - 1$

$= 25(24\lambda + 1) - 1$

$= 600\lambda + 24$

$= 24\,(25\lambda + 1)$, which is divisible by 24.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 24 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $n^3 + 2n$ is divisible by 3.

For $n = 1$: the value is 3, divisible by 3. So $P(1)$ is true.

Assume $P(k)$: $k^3 + 2k = 3\lambda$.

At $n = k + 1$: $(k+1)^3 + 2(k+1)$

$= (k^3 + 2k) + (3k^2 + 3k + 3)$

$= 3\lambda + 3(k^2 + k + 1)$, which is divisible by 3.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 3 for all $n \in \mathbb{N}$

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19. Is $38 \equiv 5 \pmod 7$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$38 - 5 = 33$

33 leaves remainder 5 on division by 7, so no.

Ans: No: $38 - 5 = 33$ is not divisible by 7

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20. 9 is odd. Using a standard result, show that $9(9^2 - 1)$ is divisible by 24 and find the quotient. [2]

For odd n, $n(n^2 - 1)$ is divisible by 24.

$9(81 - 1) = 9 \times 80 = 720 = 24 \times 30$

Ans: 30
