EXPRESSWALA ACADEMY

# Maths 1A: Mathematical Induction

## Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

**1.** For the statement $P(n)$: $6 + 11 + 16 + \cdots + (5n + 1) = \frac{n(5n+7)}{2}$, check $P(2)$. [1 mark]

**2.** To prove $1 + 9 + 17 + \cdots + (8n - 7) = n(4n - 3)$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1 mark]

**3.** Prove by mathematical induction that $2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + \cdots + (n + 1)(n + 2)(n + 3) = \frac{n(n+5)(n^2+5n+10)}{4}$ for all $n \in \mathbb{N}$. [5 marks]

**4.** Prove by mathematical induction that $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$ for all $n \in \mathbb{N}$. [5 marks]

**5.** Find $\sum_{k=1}^{99} k$. [1 mark]

**6.** Find $11^3 + 12^3 + \cdots + 20^3$. [2 marks]

**7.** Find $1^2 + 3^2 + 5^2 + \cdots + 19^2$. [2 marks]

**8.** Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 18^2$. [1 mark]

**9.** Using the standard sums, find $\sum_{k=1}^{n} k(5k - 2)$ in factorised form. [3 marks]

**10.** Find the $40$th term and the sum of the first $40$ terms of $8, 11, 14, \ldots$ [2 marks]

**11.** Find the sum to infinity of $9 + 6 + 4 + \frac{8}{3} + \ldots$ [2 marks]

**12.** Find the sum to infinity of $2 + 7 \cdot \frac{1}{3} + 12 \cdot \frac{1}{9} + 17 \cdot \frac{1}{27} + \ldots$ [3 marks]

**13.** Find the $n$th term of the sequence $2, 9, 18, 29, \ldots$ [3 marks]

**14.** Find the $n$th term of the sequence $1, 7, 25, 79, \ldots$ [3 marks]

**15.** In proving $1 \cdot 5 + 3 \cdot 7 + \cdots + (2n - 3)(2n + 1)$ (up to $(n - 1)$ terms) is equal to a formula, at which value of $n$ do we start? [1 mark]

**16.** Using its values at $n = 1$ and $n = 2$, find the largest number that divides $3^{n+1} + 5^n + 4$ for all $n \in \mathbb{N}$. [2 marks]

**17.** Prove by mathematical induction that $8^n - 5^n$ is divisible by $3$ for all $n \in \mathbb{N}$. [5 marks]

**18.** Prove by mathematical induction that $3^{2n} + 7$ is divisible by $8$ for all $n \in \mathbb{N}$. [5 marks]

**19.** Is $77 \equiv 12 \pmod{13}$? Give the reason. [1 mark]

**20.** Using a standard result, show that $9^3 + 10^3 + 11^3$ is divisible by $9$ and find the quotient. [2 marks]

## Key sheet

| Q | Answer |
|---|---|
| 1 | Both sides equal $17$, so $P(2)$ is true |
| 2 | $8k+1$ |
| 3 | Proved: true for all $n \in \mathbb{N}$ |
| 4 | Proved: true for all $n \in \mathbb{N}$ |
| 5 | $4950$ |
| 6 | $41075$ |
| 7 | $1330$ |
| 8 | $-171$ |
| 9 | $\frac{n(n+1)(10n-1)}{6}$ |
| 10 | $t_{40} = 125$, $S_{40} = 2660$ |
| 11 | $27$ |
| 12 | $\frac{27}{4}$ |
| 13 | $T_n = n^2 + 4n - 3$ |
| 14 | $T_n = 3^n - 2$ |
| 15 | $n = 2$ |
| 16 | $2$ |
| 17 | Proved: divisible by $3$ for all $n \in \mathbb{N}$ |
| 18 | Proved: divisible by $8$ for all $n \in \mathbb{N}$ |
| 19 | Yes: $77 - 12 = 65$ is divisible by $13$ |
| 20 | $340$ |

## Solutions

**1.** For the statement $P(n)$: $6 + 11 + 16 + \cdots + (5n + 1) = \frac{n(5n+7)}{2}$, check $P(2)$. [1]

$P(2)$: L.H.S. $= 6 + 11 = 17$

R.H.S. at $n = 2$: $\frac{n(5n+7)}{2} = 17$

L.H.S. $=$ R.H.S., so $P(2)$ is true.

**Ans: Both sides equal $17$, so $P(2)$ is true**

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**2.** To prove $1 + 9 + 17 + \cdots + (8n - 7) = n(4n - 3)$ by induction, which term is added to both sides of $P(k)$ to get $P(k + 1)$? [1]

The $n$th term is $(8n - 7)$.

Put $n = k + 1$: the $(k + 1)$th term is $8k + 1$.

**Ans:** $8k + 1$

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**3.** Prove by mathematical induction that $2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + \cdots + (n + 1)(n + 2)(n + 3) = \frac{n(n+5)(n^2+5n+10)}{4}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + \cdots + (n + 1)(n + 2)(n + 3) = \frac{n(n+5)(n^2+5n+10)}{4}$.

For $n = 1$: L.H.S. $= 2 \cdot 3 \cdot 4 = 24$, R.H.S. $= 24$. So $P(1)$ is true.

Assume $P(k)$ is true: $2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + \cdots + (k + 1)(k + 2)(k + 3) = \frac{k(k+5)(k^2+5k+10)}{4}$.

Add the $(k + 1)$th term $(k + 2)(k + 3)(k + 4)$ to both sides.

L.H.S. of $P(k + 1) = \frac{k(k+5)(k^2+5k+10)}{4} + (k + 2)(k + 3)(k + 4)$

$= \frac{k^4}{4} + \frac{7k^3}{2} + \frac{71k^2}{4} + \frac{77k}{2} + 24$

$= \frac{(k+1)(k+6)(k^2+7k+16)}{4}$, which is $\frac{n(n+5)(n^2+5n+10)}{4}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Ans: Proved: true for all $n \in \mathbb{N}$**

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**4.** Prove by mathematical induction that $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$.

For $n = 1$: L.H.S. $= \frac{1}{2 \cdot 5} = \frac{1}{10}$, R.H.S. $= \frac{1}{10}$. So $P(1)$ is true.

Assume $P(k)$ is true: $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$.

Add the $(k + 1)$th term $\frac{1}{(3k+2)(3k+5)}$ to both sides.

L.H.S. of $P(k + 1) = \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$

$= \frac{3k^2+5k+2}{2(3k+2)(3k+5)}$

$= \frac{k+1}{2(3k+5)}$, which is $\frac{n}{6n+4}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Ans: Proved: true for all $n \in \mathbb{N}$**

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**5.** Find $\sum_{k=1}^{99} k$. [1]

$\sum n = \frac{n(n+1)}{2}$

$n = 99$ : $\frac{99 \cdot 100}{2} = 4950$

**Ans:** $4950$

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**6.** Find $11^3 + 12^3 + \cdots + 20^3$. [2]

The sum from 1 to 20 minus the sum from 1 to 10.

$\sum_{k=1}^{20} k^3 = 44100,\ \sum_{k=1}^{10} k^3 = 3025$

$\therefore \text{sum} = 44100 - 3025 = 41075$

**Ans:** $41075$

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**7.** Find $1^2 + 3^2 + 5^2 + \cdots + 19^2$. [2]

$2n - 1 = 19 \Rightarrow n = 10$

$\frac{n}{3}(4n^2 - 1) = \frac{10}{3}(400 - 1) = 1330$

**Ans:** $1330$

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**8.** Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 18^2$. [1]

There are $n = 18$ terms and $n$ is even.

$-\frac{n(n+1)}{2} = -\frac{18 \cdot 19}{2} = -171$

**Ans:** $-171$

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**9.** Using the standard sums, find $\sum_{k=1}^{n} k(5k - 2)$ in factorised form. [3]

$k(5k - 2) = 5k^2 - 2k$

$\sum_{k=1}^{n}(5k^2 - 2k) = 5\sum k^2 - 2\sum k$

$= 5 \cdot \frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{n(n+1)}{2}$

$= \frac{10n^3 + 9n^2 - n}{6}$

$= \frac{n(n+1)(10n-1)}{6}$

**Ans:** $\frac{n(n+1)(10n-1)}{6}$

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**10.** Find the 40th term and the sum of the first 40 terms of $8, 11, 14, \ldots$ [2]

$a = 8$, $d = 3$.

$t_{40} = a + 39d = 8 + 39(3) = 125$

$S_{40} = \frac{40}{2}[a + l] = \frac{40}{2}(8 + 125) = 2660$

**Ans:** $t_{40} = 125$, $S_{40} = 2660$

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**11.** Find the sum to infinity of $9 + 6 + 4 + \frac{8}{3} + \ldots$ [2]

$a = 9$, $r = \frac{2}{3}$, $|r| < 1$.

$S_\infty = \frac{a}{1-r} = \frac{9}{1-(\frac{2}{3})} = 27$

**Ans:** $27$

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**12.** Find the sum to infinity of $2 + 7 \cdot \frac{1}{3} + 12 \cdot \frac{1}{9} + 17 \cdot \frac{1}{27} + \ldots$ [3]

This is an A.G.P. with $a = 2$, $d = 5$, $r = \frac{1}{3}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{2}{\frac{2}{3}} + \frac{\frac{5}{3}}{\frac{4}{9}}$

$= 3 + \frac{15}{4} = \frac{27}{4}$

**Ans:** $\frac{27}{4}$

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**13.** Find the $n$th term of the sequence $2, 9, 18, 29, \ldots$ [3]

Differences: $7, 9, 11$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 2,\ 4a + 2b + c = 9,\ 9a + 3b + c = 18$

$3a + b = 7,\ 5a + b = 9 \Rightarrow a = 1,\ b = 4,\ c = -3$

$\therefore T_n = n^2 + 4n - 3$ (check: $T_4 = 29$)

**Ans:** $T_n = n^2 + 4n - 3$

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**14.** Find the $n$th term of the sequence $1, 7, 25, 79, \ldots$ [3]

Differences: $6, 18, 54$, in G.P. with ratio $3$, so $T_n = A \cdot 3^n + B$.

$n = 1:\ 3A + B = 1;\ n = 2:\ 9A + B = 7$

$6A = 6 \Rightarrow A = 1,\ B = -2$

$\therefore T_n = 3^n - 2$

**Ans:** $T_n = 3^n - 2$

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**15.** In proving $1 \cdot 5 + 3 \cdot 7 + \cdots + (2n-3)(2n+1)$ (up to $(n-1)$ terms) is equal to a formula, at which value of $n$ do we start? [1]

The series has $(n-1)$ terms.

It has at least one term only when $(n-1) \geq 1$, that is $n \geq 2$.

So the verification begins with $n = 2$.

**Ans:** $n = 2$

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**16.** Using its values at $n = 1$ and $n = 2$, find the largest number that divides $3^{n+1} + 5^n + 4$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 18$

$n = 2:\ 56$

$\text{G.C.D.}(18, 56) = 2$

So $3^{n+1} + 5^n + 4$ is divisible by $2$ (induction proves it for all $n$).

**Ans:** $2$

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**17.** Prove by mathematical induction that $8^n - 5^n$ is divisible by $3$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $8^n - 5^n$ is divisible by $3$.

For $n = 1$: the value is $3 = 3 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $8^k - 5^k = 3\lambda$ for some integer $\lambda$.

Then $8^k = 3\lambda + 5^k$.

At $n = k + 1$: $8^{k+1} - 5^{k+1} = 8(8^k) - 5 \cdot 5^k$

$= 8(3\lambda + 5^k) - 5 \cdot 5^k$

$= 24\lambda + 3 \cdot 5^k$

$= 3\left(8\lambda + 5^k\right)$, which is divisible by $3$.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Ans: Proved: divisible by $3$ for all $n \in \mathbb{N}$**

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**18.** Prove by mathematical induction that $3^{2n} + 7$ is divisible by $8$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $3^{2n} + 7$ is divisible by $8$.

For $n = 1$: the value is $16 = 8 \times 2$. So $P(1)$ is true.

Assume $P(k)$: $3^{2k} + 7 = 8\lambda$ for some integer $\lambda$.

Then $3^{2k} = 8\lambda - 7$.

At $n = k + 1$: $3^{2k+2} + 7 = 9(3^{2k}) + 7$

$= 9(8\lambda - 7) + 7$

$= 72\lambda - 56$

$= 8\,(9\lambda - 7)$, which is divisible by $8$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Ans: Proved: divisible by $8$ for all $n \in \mathbb{N}$**

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**19.** Is $77 \equiv 12 \pmod{13}$? Give the reason. [1]

$x \equiv y \pmod{m}$ means $x - y$ is divisible by $m$.

$77 - 12 = 65$

$65 = 13 \times 5$, so yes.

**Ans: Yes: $77 - 12 = 65$ is divisible by $13$**

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**20.** Using a standard result, show that $9^3 + 10^3 + 11^3$ is divisible by $9$ and find the quotient. [2]

The sum of the cubes of three consecutive natural numbers is divisible by $9$.

$729 + 1000 + 1331 = 3060 = 9 \times 340$

**Ans:** $340$

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