EXPRESSWALA ACADEMY

Maths 1A: Mathematical Induction

Question Paper, Set 01

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Let $P(n)$: $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$. Verify that $P(3)$ is true. [1 mark]

2. In proving $1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + \cdots + n(n + 4) = \frac{n(n+1)(2n+13)}{6}$ by induction, what term is added to both sides of $P(k)$? [1 mark]

3. Prove by mathematical induction that $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n+1)(2n+7)}{6}$ for all $n \in \mathbb{N}$. [5 marks]

4. Prove by mathematical induction that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. [5 marks]

5. Find $1^2 + 2^2 + 3^2 + \cdots + 12^2$. [1 mark]

6. Find $1^3 + 2^3 + 3^3 + \cdots + 12^3$. [2 marks]

7. Find the sum $1^2 + 3^2 + 5^2 + \ldots$ to 8 terms. [2 marks]

8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 7^2$. [1 mark]

9. Using the standard sums, find $\sum_{k=1}^{n} k(3k + 1)$ in factorised form. [3 marks]

10. Find the 25th term and the sum of the first 25 terms of $5, 9, 13, \ldots$ [2 marks]

11. Find the 8th term and the sum of the first 8 terms of $3, 6, 12, \ldots$ [2 marks]

12. Find the sum to infinity of $2 + 5 \cdot \frac{1}{4} + 8 \cdot \frac{1}{16} + 11 \cdot \frac{1}{64} + \ldots$ [3 marks]

13. Find the nth term of the sequence $6, 13, 24, 39, \ldots$ [3 marks]

14. Find the nth term of the sequence $5, 7, 11, 19, \ldots$ [3 marks]

15. A series is written $1 \cdot 3 + 3 \cdot 5 + \cdots + (2n - 3)(2n - 1)$, that is, up to $(n - 1)$ terms. With which value of n should the verification begin? [1 mark]

16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $9^n - 4^n$ for all $n \in \mathbb{N}$. [2 marks]

17. Prove by mathematical induction that $9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5 marks]

18. Prove by mathematical induction that $4^n - 3n - 1$ is divisible by 9 for all $n \in \mathbb{N}$. [5 marks]

19. Is $47 \equiv 11 \pmod 6$? Give the reason. [1 mark]

20. Using a standard result, show that $4^3 + 5^3 + 6^3$ is divisible by 9 and find the quotient. [2 marks]

Key sheet

Q	Answer
1	Both sides equal 15, so $P(3)$ is true
2	$(k+1)(k+5)$
3	Proved: true for all $n \in \mathbb{N}$
4	Proved: true for all $n \in \mathbb{N}$
5	650
6	6084
7	680
8	28
9	$n(n+1)^2$
10	$t_{25} = 101$, $S_{25} = 1325$
11	$t_8 = 384$, $S_8 = 765$
12	4
13	$T_n = 2n^2 + n + 3$
14	$T_n = 2^n + 3$
15	$n = 2$
16	5
17	Proved: divisible by 7 for all $n \in \mathbb{N}$
18	Proved: divisible by 9 for all $n \in \mathbb{N}$
19	Yes: $47 - 11 = 36$ is divisible by 6
20	45

Solutions

1. Let $P(n)$: $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$. Verify that $P(3)$ is true. [1]

$P(3)$: L.H.S. $= 1 + 5 + 9 = 15$

R.H.S. at $n = 3$: $n(2n - 1) = 15$

L.H.S. $=$ R.H.S., so $P(3)$ is true.

Ans: Both sides equal 15, so $P(3)$ is true

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2. In proving $1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + \cdots + n(n + 4) = \frac{n(n+1)(2n+13)}{6}$ by induction, what term is added to both sides of $P(k)$? [1]

The nth term is $n(n + 4)$.

Put $n = k + 1$: the $(k + 1)$th term is $(k + 1)(k + 5)$.

Ans: $(k + 1)(k + 5)$

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3. Prove by mathematical induction that $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n+1)(2n+7)}{6}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n+1)(2n+7)}{6}$.

For $n = 1$: L.H.S. $= 1 \cdot 3 = 3$, R.H.S. $= 3$. So $P(1)$ is true.

Assume $P(k)$ is true: $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k + 2) = \frac{k(k+1)(2k+7)}{6}$.

Add the $(k + 1)$th term $(k + 1)(k + 3)$ to both sides.

L.H.S. of $P(k + 1) = \frac{k(k+1)(2k+7)}{6} + (k + 1)(k + 3)$

$= \frac{2k^3+15k^2+31k+18}{6}$

$= \frac{(k+1)(k+2)(2k+9)}{6}$, which is $\frac{n(n+1)(2n+7)}{6}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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4. Prove by mathematical induction that $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

For $n = 1$: L.H.S. $= \frac{1}{1\cdot2} = \frac{1}{2}$, R.H.S. $= \frac{1}{2}$. So $P(1)$ is true.

Assume $P(k)$ is true: $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$.

Add the $(k + 1)$th term $\frac{1}{(k+1)(k+2)}$ to both sides.

L.H.S. of $P(k + 1) = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$, which is $\frac{n}{n+1}$ with $n = k + 1$, the R.H.S. of $P(k + 1)$.

So $P(k + 1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: true for all $n \in \mathbb{N}$

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5. Find $1^2 + 2^2 + 3^2 + \cdots + 12^2$. [1]

$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

$n = 12$: $\frac{12\cdot13\cdot25}{6} = 650$

Ans: 650

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6. Find $1^3 + 2^3 + 3^3 + \cdots + 12^3$. [2]

$\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

$n = 12: \ \left(\frac{12 \cdot 13}{2}\right)^2 = 78^2 = 6084$

Ans: 6084

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7. Find the sum $1^2 + 3^2 + 5^2 + \ldots$ to 8 terms. [2]

$\sum (2k-1)^2 = \frac{n}{3}(4n^2 - 1)$

$n = 8: \ \frac{8}{3}(4 \cdot 64 - 1) = \frac{8 \cdot 255}{3} = 680$

Ans: 680

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8. Find $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 7^2$. [1]

There are $n = 7$ terms and n is odd.

$\frac{n(n+1)}{2} = \frac{7 \cdot 8}{2} = 28$

Ans: 28

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9. Using the standard sums, find $\sum_{k=1}^{n} k(3k+1)$ in factorised form. [3]

$k(3k+1) = 3k^2 + k$

$\sum_{k=1}^{n}(3k^2 + k) = 3\sum k^2 + \sum k$

$= 3 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= n^3 + 2n^2 + n$

$= n(n+1)^2$

Ans: $n(n+1)^2$

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10. Find the 25th term and the sum of the first 25 terms of $5, 9, 13, \ldots$ [2]

$a = 5$, $d = 4$.

$t_{25} = a + 24d = 5 + 24(4) = 101$

$S_{25} = \frac{25}{2}[a + l] = \frac{25}{2}(5 + 101) = 1325$

Ans: $t_{25} = 101$, $S_{25} = 1325$

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11. Find the 8th term and the sum of the first 8 terms of $3, 6, 12, \ldots$ [2]

$a = 3$, $r = 2$.

$t_8 = ar^7 = 3 \cdot 2^7 = 384$

$S_8 = a\frac{r^8 - 1}{r - 1} = 3 \cdot \frac{256 - 1}{1} = 765$

Ans: $t_8 = 384$, $S_8 = 765$

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12. Find the sum to infinity of $2 + 5 \cdot \frac{1}{4} + 8 \cdot \frac{1}{16} + 11 \cdot \frac{1}{64} + \ldots$ [3]

This is an A.G.P. with $a = 2$, $d = 3$, $r = \frac{1}{4}$, and $|r| < 1$.

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

$= \frac{2}{\frac{3}{4}} + \frac{\frac{3}{4}}{\frac{9}{16}}$

$= \frac{8}{3} + \frac{4}{3} = 4$

Ans: 4

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13. Find the nth term of the sequence $6, 13, 24, 39, \ldots$ [3]

Differences: $7, 11, 15$, in A.P., so $T_n = an^2 + bn + c$.

$a + b + c = 6,\ 4a + 2b + c = 13,\ 9a + 3b + c = 24$

$3a + b = 7,\ 5a + b = 11 \Rightarrow a = 2,\ b = 1,\ c = 3$

$\therefore T_n = 2n^2 + n + 3$ (check: $T_4 = 39$)

Ans: $T_n = 2n^2 + n + 3$

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14. Find the nth term of the sequence $5, 7, 11, 19, \ldots$ [3]

Differences: $2, 4, 8$, in G.P. with ratio 2, so $T_n = A \cdot 2^n + B$.

$n = 1:\ 2A + B = 5;\ n = 2:\ 4A + B = 7$

$2A = 2 \Rightarrow A = 1,\ B = 3$

$\therefore T_n = 2^n + 3$

Ans: $T_n = 2^n + 3$

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15. A series is written $1 \cdot 3 + 3 \cdot 5 + \cdots + (2n-3)(2n-1)$, that is, up to $(n-1)$ terms. With which value of n should the verification begin? [1]

The series has $(n-1)$ terms.

It has at least one term only when $(n-1) \geq 1$, that is $n \geq 2$.

So the verification begins with $n = 2$.

Ans: $n = 2$

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16. Using its values at $n = 1$ and $n = 2$, find the largest number that divides $9^n - 4^n$ for all $n \in \mathbb{N}$. [2]

$n = 1:\ 5$

$n = 2:\ 65$

$\text{G.C.D.}(5, 65) = 5$

So $9^n - 4^n$ is divisible by 5 (induction proves it for all n).

Ans: 5

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17. Prove by mathematical induction that $9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $9^n - 2^n$ is divisible by 7.

For $n = 1$: the value is $7 = 7 \times 1$. So $P(1)$ is true.

Assume $P(k)$: $9^k - 2^k = 7\lambda$ for some integer λ.

Then $9^k = 7\lambda + 2^k$.

At $n = k + 1$: $9^{k+1} - 2^{k+1} = 9(9^k) - 2 \cdot 2^k$

$= 9(7\lambda + 2^k) - 2 \cdot 2^k$

$= 63\lambda + 7 \cdot 2^k$

$= 7\left(9\lambda + 2^k\right)$, which is divisible by 7.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 7 for all $n \in \mathbb{N}$

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18. Prove by mathematical induction that $4^n - 3n - 1$ is divisible by 9 for all $n \in \mathbb{N}$. [5]

Let $P(n)$: $4^n - 3n - 1$ is divisible by 9.

For $n = 1$: $4 - 3 - 1 = 0$, which is divisible by 9. So $P(1)$ is true.

Assume $P(k)$: $4^k - 3k - 1 = 9\lambda$, so $4^k = 9\lambda + 3k + 1$.

At $n = k + 1$: $4^{k+1} - 3(k+1) - 1 = 4 \cdot 4^k - 3k - 4$

$= 4(9\lambda + 3k + 1) - 3k - 4$

$= 36\lambda + 12k + 4 - 3k - 4$

$= 36\lambda + 9k = 9(4\lambda + k)$, which is divisible by 9.

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ans: Proved: divisible by 9 for all $n \in \mathbb{N}$

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19. Is $47 \equiv 11 \pmod 6$? Give the reason. [1]

$x \equiv y \pmod m$ means $x - y$ is divisible by m.

$47 - 11 = 36$

$36 = 6 \times 6$, so yes.

Ans: Yes: $47 - 11 = 36$ is divisible by 6

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20. Using a standard result, show that $4^3 + 5^3 + 6^3$ is divisible by 9 and find the quotient. [2]

The sum of the cubes of three consecutive natural numbers is divisible by 9.

$64 + 125 + 216 = 405 = 9 \times 45$

Ans: 45
