EXPRESSWALA ACADEMY

# Maths 1A: Transformations

## Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

**1.** Write $2\cos 8\theta \sin 4\theta$ as a sum or difference of two trigonometric ratios. [1 mark]

**2.** Evaluate $4\cos 165^\circ \sin 15^\circ$ by changing the product into a sum. [2 marks]

**3.** Write $\cos 3\theta - \cos\theta$ in the form of a product. [1 mark]

**4.** Simplify $\sin 67^\circ - \sin 53^\circ$. [1 mark]

**5.** Evaluate $\cos 15^\circ - \cos 45^\circ$ using a sum to product formula. [2 marks]

**6.** Show that $\sin 54^\circ + \sin 6^\circ - \sin 114^\circ = 0$. [3 marks]

**7.** Find the value of $\frac{\sin 67^\circ + \sin 53^\circ}{\cos 67^\circ + \cos 53^\circ}$. [1 mark]

**8.** Simplify $\frac{\sin 10\theta - \sin 2\theta}{\cos 2\theta - \cos 10\theta}$. [2 marks]

**9.** Simplify $\frac{\sin 5\theta + \sin 7\theta + \sin 3\theta}{\cos 5\theta + \cos 7\theta + \cos 3\theta}$. [3 marks]

**10.** Show that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$. [5 marks]

**11.** In a triangle $ABC$, express $\tan\frac{A+B}{2}$ in terms of $C$. [1 mark]

**12.** Verify $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$ for the triangle with $A = 30^\circ$, $B = 60^\circ$, $C = 90^\circ$. [2 marks]

**13.** In a triangle $ABC$, prove that $\cos^2 A - \cos^2 B + \cos^2 C = 1 - 2\sin A \cos B \sin C$. [5 marks]

**14.** If $A + B + C = 180^\circ$, prove that $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$. [5 marks]

**15.** If $\cos x - \cos y = \frac{5}{4}$ and $\sin x + \sin y = \frac{2}{3}$, find $\tan\frac{x-y}{2}$. [2 marks]

**16.** If $\cos x - \cos y = \frac{4}{5}$ and $\sin x + \sin y = \frac{2}{7}$, find $\sin(x-y)$ and $\cos(x-y)$. [3 marks]

**17.** If $\cos x + \cos y = \frac{1}{4}$ and $\sin x + \sin y = \frac{7}{5}$, find $\tan\frac{x+y}{2}$, $\sin(x+y)$, $\cos(x+y)$ and $\tan(x+y)$. [5 marks]

**18.** If $\cos x + \cos y = \frac{4}{5}$ and $\sin x - \sin y = \frac{6}{5}$, find $\cos(x+y)$. [2 marks]

**19.** If $\cos x + \cos y = \frac{5}{6}$ and $\sin x + \sin y = \frac{3}{2}$, find $\cos(x-y)$ and $\cos\frac{x-y}{2}$. [3 marks]

**20.** In a triangle $ABC$, $A = 45^\circ$ and $B = 60^\circ$. Find $\cos A + \cos B + \cos C$ and check that it does not exceed $\frac{3}{2}$. [1 mark]

## Key sheet

| Q | Answer |
|---|---|
| 1 | $\sin 12\theta - \sin 4\theta$ |
| 2 | $-1$ |
| 3 | $-2\sin 2\theta \sin\theta$ |
| 4 | $\sin 7^\circ$ |
| 5 | $\frac{-\sqrt{2}+\sqrt{6}}{4}$ |
| 6 | Shown: the expression is $0$ |
| 7 | $\sqrt{3}$ |
| 8 | $\cot 6\theta$ |
| 9 | $\tan 5\theta$ |
| 10 | Shown: $\frac{1}{8}$ |
| 11 | $\cot\frac{C}{2}$ |
| 12 | Both sides equal $1$ |
| 13 | Proved |
| 14 | Proved |
| 15 | $\tan\frac{x-y}{2} = -\frac{15}{8}$ |
| 16 | $\sin(x-y) = -\frac{140}{221},\ \cos(x-y) = -\frac{171}{221}$ |
| 17 | $\tan\frac{x+y}{2} = \frac{28}{5},\ \sin(x+y) = \frac{280}{809},\ \cos(x+y) = -\frac{759}{809},\ \tan(x+y) = -\frac{280}{759}$ |
| 18 | $\cos(x+y) = \frac{1}{25}$ |
| 19 | $\cos(x-y) = \frac{17}{36},\ \cos\frac{x-y}{2} = \pm\frac{\sqrt{106}}{12}$ |
| 20 | $\frac{1}{2} + \frac{\sqrt{2}+\sqrt{6}}{4}$ |

## Solutions

**1.** Write $2\cos 8\theta \sin 4\theta$ as a sum or difference of two trigonometric ratios. [1]

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$ with $A = 8\theta,\ B = 4\theta$.

$2\cos 8\theta \sin 4\theta = \sin 12\theta - \sin 4\theta$

**Ans:** $\sin 12\theta - \sin 4\theta$

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**2.** Evaluate $4\cos 165^\circ \sin 15^\circ$ by changing the product into a sum. [2]

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$4\cos 165^\circ \sin 15^\circ = 2\left(\sin 180^\circ - \sin 150^\circ\right)$

$= 2\left(0 - \frac{1}{2}\right) = -1$

**Ans:** $-1$

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**3.** Write $\cos 3\theta - \cos\theta$ in the form of a product. [1]

$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos 3\theta - \cos\theta = -2\sin 2\theta \sin\theta$

**Ans:** $-2\sin 2\theta\sin\theta$

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**4.** Simplify $\sin 67^\circ - \sin 53^\circ$. [1]

The half sum of the angles is $60^\circ$, a standard angle.

$\sin 67^\circ - \sin 53^\circ = 2\cos 60^\circ \sin 7^\circ = \sin 7^\circ$

**Ans:** $\sin 7^\circ$

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**5.** Evaluate $\cos 15^\circ - \cos 45^\circ$ using a sum to product formula. [2]

$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos 15^\circ - \cos 45^\circ = 2\sin 30^\circ \sin 15^\circ$

$= \frac{-\sqrt{2}+\sqrt{6}}{4}$

**Ans:** $\frac{-\sqrt{2}+\sqrt{6}}{4}$

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**6.** Show that $\sin 54^\circ + \sin 6^\circ - \sin 114^\circ = 0$. [3]

$\sin 6^\circ + \sin 54^\circ = 2\sin\frac{6^\circ+54^\circ}{2}\cos\frac{6^\circ-54^\circ}{2} = 2\sin 30^\circ\cos -24^\circ = \cos -24^\circ$

$\sin 114^\circ = \sin(90^\circ - -24^\circ) = \cos -24^\circ$

$\therefore \sin 54^\circ + \sin 6^\circ - \sin 114^\circ = \cos -24^\circ - \cos -24^\circ = 0$

**Ans: Shown: the expression is** $0$

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**7.** Find the value of $\frac{\sin 67^\circ + \sin 53^\circ}{\cos 67^\circ + \cos 53^\circ}$. [1]

$\frac{\sin 67^\circ+\sin 53^\circ}{\cos 67^\circ+\cos 53^\circ} = \frac{2\sin 60^\circ\cos 7^\circ}{2\cos 60^\circ\cos 7^\circ}$

$= \tan 60^\circ = \sqrt{3}$

**Ans:** $\sqrt{3}$

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**8.** Simplify $\frac{\sin 10\theta - \sin 2\theta}{\cos 2\theta - \cos 10\theta}$. [2]

$\frac{\sin 10\theta - \sin 2\theta}{\cos 2\theta - \cos 10\theta} = \frac{2\cos 6\theta \sin 4\theta}{2\sin 6\theta \sin 4\theta}$

$= \cot 6\theta$

**Ans:** $\cot 6\theta$

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**9.** Simplify $\frac{\sin 5\theta + \sin 7\theta + \sin 3\theta}{\cos 5\theta + \cos 7\theta + \cos 3\theta}$. [3]

$\sin 7\theta + \sin 3\theta = 2\sin 5\theta \cos 2\theta$, $\cos 7\theta + \cos 3\theta = 2\cos 5\theta \cos 2\theta$

$\text{Numerator} = \sin 5\theta(2\cos 2\theta + 1)$, $\text{Denominator} = \cos 5\theta(2\cos 2\theta + 1)$

Cancel the common factor $2\cos 2\theta + 1$: the fraction is $\tan 5\theta$.

**Ans:** $\tan 5\theta$

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**10.** Show that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$. [5]

$\cos 40^\circ \cos 80^\circ = \frac{1}{2}[\cos 120^\circ + \cos 40^\circ] = \frac{1}{2}\cos 40^\circ - \frac{1}{4}$

$\cos 20^\circ \left(\frac{1}{2}\cos 40^\circ - \frac{1}{4}\right) = \frac{1}{4}(2\cos 40^\circ \cos 20^\circ) - \frac{1}{4}\cos 20^\circ$

$= \frac{1}{4}[\cos 60^\circ + \cos 20^\circ] - \frac{1}{4}\cos 20^\circ = \frac{1}{4}\cos 60^\circ$

$= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

**Ans: Shown:** $\frac{1}{8}$

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**11.** In a triangle $ABC$, express $\tan \frac{A+B}{2}$ in terms of $C$. [1]

In a triangle, $A + B = 180^\circ - C$.

$\tan \frac{A+B}{2} = \cot \frac{C}{2}$

**Ans:** $\cot \frac{C}{2}$

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**12.** Verify $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$ for the triangle with $A = 30^\circ$, $B = 60^\circ$, $C = 90^\circ$. [2]

LHS $= 1$

RHS $= 1$

Both sides equal 1, so the identity holds.

**Ans: Both sides equal** $1$

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**13.** In a triangle $ABC$, prove that $\cos^2 A - \cos^2 B + \cos^2 C = 1 - 2\sin A \cos B \sin C$. [5]

$A + B + C = 180^\circ$, so $A + B = 180^\circ - C$ and similar.

$\cos^2 A + \cos^2 C = 1 + \frac{1}{2}(\cos 2A + \cos 2C) = 1 + \cos(A + C)\cos(A - C) = 1 - \cos B \cos(A - C)$

$-\cos^2 B = -\cos B \cos B = \cos B \cos(A + C)$

$\text{LHS} = 1 - \cos B[\cos(A - C) - \cos(A + C)] = 1 - \cos B \cdot 2\sin A \sin C$

$= 1 - 2\sin A \cos B \sin C = \text{RHS}$

**Ans: Proved**

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**14.** If $A+B+C=180^\circ$, prove that $\cos 2A-\cos 2B+\cos 2C=1-4\sin A\cos B\sin C$. [5]

$A+B+C=180^\circ$, so $A+B=180^\circ-C$ and similar.

$\cos 2A+\cos 2C=2\cos(A+C)\cos(A-C)=-2\cos B\cos(A-C)$

$-\cos 2B=1-2\cos^2 B=1+2\cos B\cos(A+C)$

$\text{LHS}=1-2\cos B[\cos(A-C)-\cos(A+C)]=1-2\cos B\cdot 2\sin A\sin C$

$=1-4\sin A\cos B\sin C=\text{RHS}$

**Ans: Proved**

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**15.** If $\cos x-\cos y=\frac{5}{4}$ and $\sin x+\sin y=\frac{2}{3}$, find $\tan\frac{x-y}{2}$. [2]

Transforming: $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}=\frac{5}{4}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{2}{3}$.

Dividing: $\tan\frac{x-y}{2}=-\frac{15}{8}$

**Ans:** $\tan\frac{x-y}{2}=-\frac{15}{8}$

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**16.** If $\cos x-\cos y=\frac{4}{5}$ and $\sin x+\sin y=\frac{2}{7}$, find $\sin(x-y)$ and $\cos(x-y)$. [3]

Transforming: $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}=\frac{4}{5}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{2}{7}$.

Dividing: $\tan\frac{x-y}{2}=-\frac{14}{5}$

With $t=-\frac{14}{5}$: $\sin(x-y)=\frac{2t}{1+t^2}=-\frac{140}{221}$, $\cos(x-y)=\frac{1-t^2}{1+t^2}=-\frac{171}{221}$

**Ans:** $\sin(x-y)=-\frac{140}{221}$, $\cos(x-y)=-\frac{171}{221}$

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**17.** If $\cos x+\cos y=\frac{1}{4}$ and $\sin x+\sin y=\frac{7}{5}$, find $\tan\frac{x+y}{2}$, $\sin(x+y)$, $\cos(x+y)$ and $\tan(x+y)$. [5]

Transforming: $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{1}{4}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{7}{5}$.

Dividing: $\tan\frac{x+y}{2}=\frac{28}{5}$

With $t=\frac{28}{5}$: $\sin(x+y)=\frac{2t}{1+t^2}=\frac{280}{809}$, $\cos(x+y)=\frac{1-t^2}{1+t^2}=-\frac{759}{809}$

$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=-\frac{280}{759}$

**Ans:** $\tan\frac{x+y}{2}=\frac{28}{5}$, $\sin(x+y)=\frac{280}{809}$, $\cos(x+y)=-\frac{759}{809}$, $\tan(x+y)=-\frac{280}{759}$

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**18.** If $\cos x+\cos y=\frac{4}{5}$ and $\sin x-\sin y=\frac{6}{5}$, find $\cos(x+y)$. [2]

Square both and add, using $\cos^2+\sin^2=1$:

$a^2+b^2=2+2(\cos x\cos y-\sin x\sin y)=2+2\cos(x+y)$

$\frac{16}{25}+\frac{36}{25}=2+2\cos(x+y)$

$\cos(x+y)=\frac{1}{25}$

**Ans:** $\cos(x+y)=\frac{1}{25}$

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**19.** If $\cos x+\cos y=\frac{5}{6}$ and $\sin x+\sin y=\frac{3}{2}$, find $\cos(x-y)$ and $\cos\frac{x-y}{2}$. [3]

Square both and add, using $\cos^2+\sin^2=1$:

$a^2+b^2=2+2(\cos x\cos y+\sin x\sin y)=2+2\cos(x-y)$

$\frac{25}{36}+\frac{9}{4}=2+2\cos(x-y)$

$\cos(x-y)=\frac{17}{36}$

$\cos^2\frac{x-y}{2}=\frac{1+\cos(x-y)}{2}=\frac{53}{72}$

$\cos\frac{x-y}{2}=\pm\frac{\sqrt{106}}{12}$

**Ans:** $\cos(x-y)=\frac{17}{36}$, $\cos\frac{x-y}{2}=\pm\frac{\sqrt{106}}{12}$

**20.** In a triangle $ABC$, $A = 45^\circ$ and $B = 60^\circ$. Find $\cos A + \cos B + \cos C$ and check that it does not exceed $\frac{3}{2}$. [1]

$C = 180^\circ - 45^\circ - 60^\circ = 75^\circ$

$\cos 45^\circ + \cos 60^\circ + \cos 75^\circ = \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{-\sqrt{2}+\sqrt{6}}{4} = \frac{1}{2} + \frac{\sqrt{2}+\sqrt{6}}{4}$

$\frac{1}{2} + \frac{\sqrt{2}+\sqrt{6}}{4} \approx 1.466 \leq 1.5$

**Ans:** $\frac{1}{2} + \frac{\sqrt{2}+\sqrt{6}}{4}$