EXPRESSWALA ACADEMY

Maths 1A: Transformations

Question Paper, Set 05

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Express $2\cos 9\theta \cos \theta$ as a sum or difference. [1 mark]

2. Find the value of $2\sin 75^\circ \sin 45^\circ$. [2 marks]

3. Write $\cos 9\theta - \cos 7\theta$ in the form of a product. [1 mark]

4. Simplify $\cos 63^\circ + \cos 57^\circ$. [1 mark]

5. Find the value of $\cos 105^\circ - \cos 15^\circ$. [2 marks]

6. Show that $\sin 99^\circ - \sin 39^\circ + \cos 111^\circ = 0$. [3 marks]

7. Evaluate $\dfrac{\sin 103^\circ + \sin 17^\circ}{\cos 103^\circ + \cos 17^\circ}$. [1 mark]

8. Show that $\dfrac{\cos 5\theta - \cos 9\theta}{\sin 9\theta + \sin 5\theta} = \tan 2\theta$. [2 marks]

9. Simplify $\dfrac{\sin 2\theta + \sin 3\theta + \sin \theta}{\cos 2\theta + \cos 3\theta + \cos \theta}$. [3 marks]

10. Show that $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \dfrac{1}{16}$. [5 marks]

11. In a triangle ABC, express $\cos \dfrac{C+A}{2}$ in terms of B. [1 mark]

12. Verify $\cos A + \cos B - \cos C = -1 + 4\cos \dfrac{A}{2} \cos \dfrac{B}{2} \sin \dfrac{C}{2}$ for the triangle with $A = 30^\circ$, $B = 90^\circ$, $C = 60^\circ$. [2 marks]

13. If $A + B + C = 180^\circ$, prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$. [5 marks]

14. If $A + B + C = 180^\circ$, prove that $\cos^2 \dfrac{A}{2} + \cos^2 \dfrac{B}{2} + \cos^2 \dfrac{C}{2} = 2 + 2\sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$. [5 marks]

15. If $\cos x + \cos y = \dfrac{7}{5}$ and $\sin x + \sin y = \dfrac{3}{4}$, find $\tan \dfrac{x+y}{2}$. [2 marks]

16. If $\cos x + \cos y = \dfrac{3}{4}$ and $\sin x - \sin y = \dfrac{7}{5}$, find $\sin(x - y)$ and $\cos(x - y)$. [3 marks]

17. If $\cos x - \cos y = \dfrac{4}{5}$ and $\sin x + \sin y = \dfrac{1}{6}$, find $\tan \dfrac{x-y}{2}$, $\sin(x - y)$, $\cos(x - y)$ and $\tan(x - y)$. [5 marks]

18. If $\cos x - \cos y = \dfrac{1}{5}$ and $\sin x + \sin y = \dfrac{5}{4}$, find $\cos(x + y)$. [2 marks]

19. If $\cos x + \cos y = \dfrac{1}{6}$ and $\sin x + \sin y = \dfrac{3}{4}$, find $\cos(x - y)$ and $\cos \dfrac{x-y}{2}$. [3 marks]

20. In a triangle ABC, $A = 120^\circ$ and $B = 30^\circ$. Find $\cos A + \cos B + \cos C$ and check that it does not exceed $\frac{3}{2}$. [1 mark]

Key sheet

Q	Answer
1	$\cos 10\theta + \cos 8\theta$
2	$\frac{1}{2} + \frac{\sqrt{3}}{2}$
3	$-2\sin 8\theta \sin \theta$
4	$\cos 3^\circ$
5	$-\frac{\sqrt{6}}{2}$
6	Shown: the expression is 0
7	$\sqrt{3}$
8	$\tan 2\theta$
9	$\tan 2\theta$
10	Shown: $\frac{1}{16}$
11	$\sin \frac{B}{2}$
12	Both sides equal $-\frac{1}{2} + \frac{\sqrt{3}}{2}$
13	Proved
14	Proved
15	$\tan \frac{x+y}{2} = \frac{15}{28}$
16	$\sin(x-y) = \frac{840}{1009},\ \cos(x-y) = -\frac{559}{1009}$
17	$\tan \frac{x-y}{2} = -\frac{24}{5},\ \sin(x-y) = -\frac{240}{601},\ \cos(x-y) = -\frac{551}{601},\ \tan(x-y) = \frac{240}{551}$
18	$\cos(x+y) = \frac{159}{800}$
19	$\cos(x-y) = -\frac{203}{288},\ \cos \frac{x-y}{2} = \pm\frac{\sqrt{85}}{24}$
20	$-\frac{1}{2} + \sqrt{3}$

Solutions

1. Express $2\cos 9\theta\cos\theta$ as a sum or difference. [1]

$2\cos A\cos B = \cos(A+B)+\cos(A-B)$ with $A = 9\theta,\ B = \theta$.

$2\cos 9\theta\cos\theta = \cos 10\theta + \cos 8\theta$

Ans: $\cos 10\theta + \cos 8\theta$

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2. Find the value of $2\sin 75^\circ \sin 45^\circ$. [2]

$2\sin A\sin B = \cos(A-B)-\cos(A+B)$

$2\sin 75^\circ \sin 45^\circ = \cos 30^\circ - \cos 120^\circ$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2}$

Ans: $\frac{1}{2} + \frac{\sqrt{3}}{2}$

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3. Write $\cos 9\theta - \cos 7\theta$ in the form of a product. [1]

$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos 9\theta - \cos 7\theta = -2\sin 8\theta\sin\theta$

Ans: $-2\sin 8\theta\sin\theta$

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4. Simplify $\cos 63^\circ + \cos 57^\circ$. [1]

The half sum of the angles is 60°, a standard angle.

$\cos 63^\circ + \cos 57^\circ = 2\cos 60^\circ\cos 3^\circ = \cos 3^\circ$

Ans: $\cos 3^\circ$

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5. Find the value of $\cos 105^\circ - \cos 15^\circ$. [2]

$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos 105^\circ - \cos 15^\circ = -2\sin 60^\circ\sin 45^\circ$

$= -\frac{\sqrt{6}}{2}$

Ans: $-\frac{\sqrt{6}}{2}$

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6. Show that $\sin 99^\circ - \sin 39^\circ + \cos 111^\circ = 0$. [3]

$\sin 99^\circ - \sin 39^\circ = 2\cos\frac{99^\circ+39^\circ}{2}\sin\frac{99^\circ-39^\circ}{2} = 2\cos 69^\circ\sin 30^\circ = \cos 69^\circ$

$\cos 111^\circ = \cos(180^\circ - 69^\circ) = -\cos 69^\circ$

$\therefore \sin 99^\circ - \sin 39^\circ + \cos 111^\circ = \cos 69^\circ - \cos 69^\circ = 0$

Ans: Shown: the expression is 0

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7. Evaluate $\frac{\sin 103^\circ+\sin 17^\circ}{\cos 103^\circ+\cos 17^\circ}$. [1]

$\frac{\sin 103^\circ+\sin 17^\circ}{\cos 103^\circ+\cos 17^\circ} = \frac{2\sin 60^\circ\cos 43^\circ}{2\cos 60^\circ\cos 43^\circ}$

$= \tan 60^\circ = \sqrt{3}$

Ans: $\sqrt{3}$

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8. Show that $\frac{\cos 5\theta - \cos 9\theta}{\sin 9\theta + \sin 5\theta} = \tan 2\theta$. [2]

$\frac{\cos 5\theta - \cos 9\theta}{\sin 9\theta + \sin 5\theta} = \frac{2\sin 7\theta \sin 2\theta}{2\sin 7\theta \cos 2\theta}$

$= \tan 2\theta$

Ans: $\tan 2\theta$

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9. Simplify $\frac{\sin 2\theta + \sin 3\theta + \sin \theta}{\cos 2\theta + \cos 3\theta + \cos \theta}$. [3]

$\sin 3\theta + \sin \theta = 2\sin 2\theta \cos \theta$, $\cos 3\theta + \cos \theta = 2\cos 2\theta \cos \theta$

$\text{Numerator} = \sin 2\theta(2\cos \theta + 1)$, $\text{Denominator} = \cos 2\theta(2\cos \theta + 1)$

Cancel the common factor $2\cos \theta + 1$: the fraction is $\tan 2\theta$.

Ans: $\tan 2\theta$

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10. Show that $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$. [5]

$\sin 30^\circ = \frac{1}{2}$, so it is a constant factor.

$\sin 50^\circ \sin 70^\circ = \frac{1}{2}\cos 20^\circ + \frac{1}{4}$

Multiplying by $\sin 10^\circ$ and using product to sum again: $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{4}\sin 30^\circ = \frac{1}{8}$

$\therefore \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$

Ans: Shown: $\frac{1}{16}$

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11. In a triangle ABC, express $\cos \frac{C+A}{2}$ in terms of B. [1]

In a triangle, $C + A = 180^\circ - B$.

$\cos \frac{C+A}{2} = \sin \frac{B}{2}$

Ans: $\sin \frac{B}{2}$

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12. Verify $\cos A + \cos B - \cos C = -1 + 4\cos \frac{A}{2}\cos \frac{B}{2}\sin \frac{C}{2}$ for the triangle with $A = 30^\circ$, $B = 90^\circ$, $C = 60^\circ$. [2]

$\text{LHS} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

$\text{RHS} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

Both sides equal $-\frac{1}{2} + \frac{\sqrt{3}}{2}$, so the identity holds.

Ans: Both sides equal $-\frac{1}{2} + \frac{\sqrt{3}}{2}$

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13. If $A + B + C = 180^\circ$, prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$. [5]

$A + B + C = 180^\circ$, so $A + B = 180^\circ - C$ and similar.

$\sin^2 A + \sin^2 B = 1 - \frac{1}{2}(\cos 2A + \cos 2B) = 1 - \cos(A+B)\cos(A-B) = 1 + \cos C \cos(A-B)$

$-\sin^2 C = -1 + \cos^2 C = -1 - \cos C \cos(A+B)$

$\text{LHS} = \cos C[\cos(A-B) - \cos(A+B)] = \cos C \cdot 2\sin A \sin B$

$= 2\sin A \sin B \cos C = \text{RHS}$

Ans: Proved

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14. If $A+B+C=180^\circ$, prove that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. [5]

$A+B+C=180^\circ$, so $A+B=180^\circ-C$ and similar.

$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}=1+\frac{1}{2}(\cos A+\cos B)=1+\sin\frac{C}{2}\cos\frac{A-B}{2}$

$\cos^2\frac{C}{2}=1-\sin^2\frac{C}{2}=1-\sin\frac{C}{2}\cos\frac{A+B}{2}$

$\text{LHS}=2+\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]=2+\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}$

$=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\text{RHS}$

Ans: Proved

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15. If $\cos x+\cos y=\frac{7}{5}$ and $\sin x+\sin y=\frac{3}{4}$, find $\tan\frac{x+y}{2}$. [2]

Transforming: $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{7}{5}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{3}{4}$.

Dividing: $\tan\frac{x+y}{2}=\frac{15}{28}$

Ans: $\tan\frac{x+y}{2}=\frac{15}{28}$

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16. If $\cos x+\cos y=\frac{3}{4}$ and $\sin x-\sin y=\frac{7}{5}$, find $\sin(x-y)$ and $\cos(x-y)$. [3]

Transforming: $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{3}{4}$ and $2\cos\frac{x+y}{2}\sin\frac{x-y}{2}=\frac{7}{5}$.

Dividing: $\tan\frac{x-y}{2}=\frac{28}{15}$

With $t=\frac{28}{15}$: $\sin(x-y)=\frac{2t}{1+t^2}=\frac{840}{1009}$, $\cos(x-y)=\frac{1-t^2}{1+t^2}=-\frac{559}{1009}$

Ans: $\sin(x-y)=\frac{840}{1009}$, $\cos(x-y)=-\frac{559}{1009}$

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17. If $\cos x-\cos y=\frac{4}{5}$ and $\sin x+\sin y=\frac{1}{6}$, find $\tan\frac{x-y}{2}$, $\sin(x-y)$, $\cos(x-y)$ and $\tan(x-y)$. [5]

Transforming: $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}=\frac{4}{5}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{1}{6}$.

Dividing: $\tan\frac{x-y}{2}=-\frac{24}{5}$

With $t=-\frac{24}{5}$: $\sin(x-y)=\frac{2t}{1+t^2}=-\frac{240}{601}$, $\cos(x-y)=\frac{1-t^2}{1+t^2}=-\frac{551}{601}$

$\tan(x-y)=\frac{\sin(x-y)}{\cos(x-y)}=\frac{240}{551}$

Ans: $\tan\frac{x-y}{2}=-\frac{24}{5}$, $\sin(x-y)=-\frac{240}{601}$, $\cos(x-y)=-\frac{551}{601}$, $\tan(x-y)=\frac{240}{551}$

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18. If $\cos x-\cos y=\frac{1}{5}$ and $\sin x+\sin y=\frac{5}{4}$, find $\cos(x+y)$. [2]

Square both and add, using $\cos^2+\sin^2=1$:

$a^2+b^2=2-2(\cos x\cos y-\sin x\sin y)=2-2\cos(x+y)$

$\frac{1}{25}+\frac{25}{16}=2-2\cos(x+y)$

$\cos(x+y)=\frac{159}{800}$

Ans: $\cos(x+y)=\frac{159}{800}$

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19. If $\cos x+\cos y=\frac{1}{6}$ and $\sin x+\sin y=\frac{3}{4}$, find $\cos(x-y)$ and $\cos\frac{x-y}{2}$. [3]

Square both and add, using $\cos^2+\sin^2=1$:

$a^2+b^2=2+2(\cos x\cos y+\sin x\sin y)=2+2\cos(x-y)$

$\frac{1}{36}+\frac{9}{16}=2+2\cos(x-y)$

$\cos(x-y)=-\frac{203}{288}$

$\cos^2\frac{x-y}{2}=\frac{1+\cos(x-y)}{2}=\frac{85}{576}$

$\cos\frac{x-y}{2}=\pm\frac{\sqrt{85}}{24}$

Ans: $\cos(x-y)=-\frac{203}{288}$, $\cos\frac{x-y}{2}=\pm\frac{\sqrt{85}}{24}$

20. In a triangle ABC, $A = 120^\circ$ and $B = 30^\circ$. Find $\cos A + \cos B + \cos C$ and check that it does not exceed $\frac{3}{2}$. [1]

$C = 180^\circ - 120^\circ - 30^\circ = 30^\circ$

$\cos 120^\circ + \cos 30^\circ + \cos 30^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = -\frac{1}{2} + \sqrt{3}$

$-\frac{1}{2} + \sqrt{3} \approx 1.232 \leq 1.5$

Ans: $-\frac{1}{2} + \sqrt{3}$
