EXPRESSWALA ACADEMY

Maths 1A: Transformations

Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Express $2\sin 7\theta\cos 2\theta$ as a sum or difference. [1 mark]

2. Evaluate $4\sin 165^\circ \sin 15^\circ$ by changing the product into a sum. [2 marks]

3. Write $\cos 3\theta - \cos 7\theta$ in the form of a product. [1 mark]

4. Simplify $\sin 41^\circ + \sin 19^\circ$. [1 mark]

5. Evaluate $\cos 135^\circ + \cos 105^\circ$ using a sum to product formula. [2 marks]

6. Show that $\cos 49^\circ + \cos 71^\circ + \cos 169^\circ = 0$. [3 marks]

7. Evaluate $\dfrac{\sin 82^\circ - \sin 8^\circ}{\cos 8^\circ - \cos 82^\circ}$. [1 mark]

8. Show that $\dfrac{\sin 8\theta - \sin 4\theta}{\cos 8\theta + \cos 4\theta} = \tan 2\theta$. [2 marks]

9. Prove that $\dfrac{\sin 7\theta + \sin 6\theta + \sin 8\theta}{\cos 7\theta + \cos 6\theta + \cos 8\theta} = \tan 7\theta$. [3 marks]

10. Show that $\tan 10^\circ \tan 50^\circ \tan 70^\circ = \dfrac{\sqrt{3}}{3}$. [5 marks]

11. In a triangle ABC, express $\sin(2B + 2C)$ in terms of A. [1 mark]

12. Verify $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$ for the triangle with $A = 120^\circ$, $B = 30^\circ$, $C = 30^\circ$. [2 marks]

13. In a triangle ABC, prove that $-\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\sin B\sin C$. [5 marks]

14. In a triangle ABC, prove that $\sin^2\dfrac{A}{2} + \sin^2\dfrac{B}{2} + \sin^2\dfrac{C}{2} = 1 - 2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$. [5 marks]

15. If $\cos x + \cos y = \dfrac{1}{3}$ and $\sin x + \sin y = \dfrac{3}{7}$, find $\tan\dfrac{x+y}{2}$. [2 marks]

16. If $\cos x + \cos y = \dfrac{1}{6}$ and $\sin x - \sin y = \dfrac{1}{4}$, find $\sin(x - y)$ and $\cos(x - y)$. [3 marks]

17. If $\cos x - \cos y = \dfrac{1}{2}$ and $\sin x + \sin y = \dfrac{2}{3}$, find $\tan\dfrac{x-y}{2}$, $\sin(x - y)$, $\cos(x - y)$ and $\tan(x - y)$. [5 marks]

18. If $\cos x + \cos y = \dfrac{4}{3}$ and $\sin x - \sin y = \dfrac{2}{7}$, find $\cos(x + y)$. [2 marks]

19. If $\cos x + \cos y = \dfrac{4}{3}$ and $\sin x + \sin y = \dfrac{2}{3}$, find $\cos(x - y)$ and $\cos\dfrac{x-y}{2}$. [3 marks]

20. In a triangle ABC, $A = 30^\circ$ and $B = 45^\circ$. Find $\cos A + \cos B + \cos C$ and check that it does not exceed $\frac{3}{2}$. [1 mark]

Key sheet

Q	Answer
1	$\sin 9\theta + \sin 5\theta$
2	$2 - \sqrt{3}$
3	$2\sin 5\theta \sin 2\theta$
4	$\cos 11^\circ$
5	$\frac{-\sqrt{6}-\sqrt{2}}{4}$
6	Shown: the expression is 0
7	1
8	$\tan 2\theta$
9	$\tan 7\theta$
10	Shown: $\frac{\sqrt{3}}{3}$
11	$-\sin 2A$
12	Both sides equal $\frac{1}{2}$
13	Proved
14	Proved
15	$\tan\frac{x+y}{2} = \frac{9}{7}$
16	$\sin(x-y) = \frac{12}{13},\ \cos(x-y) = -\frac{5}{13}$
17	$\tan\frac{x-y}{2} = -\frac{3}{4},\ \sin(x-y) = -\frac{24}{25},\ \cos(x-y) = \frac{7}{25},\ \tan(x-y) = -\frac{24}{7}$
18	$\cos(x+y) = -\frac{31}{441}$
19	$\cos(x-y) = \frac{1}{9},\ \cos\frac{x-y}{2} = \pm\frac{\sqrt{5}}{3}$
20	$\frac{-\sqrt{6}+2\sqrt{3}+3\sqrt{2}}{4}$

Solutions

1. Express $2\sin 7\theta\cos 2\theta$ as a sum or difference. [1]

$2\sin A\cos B = \sin(A+B)+\sin(A-B)$ with $A = 7\theta,\ B = 2\theta$.

$2\sin 7\theta\cos 2\theta = \sin 9\theta + \sin 5\theta$

Ans: $\sin 9\theta + \sin 5\theta$

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2. Evaluate $4\sin 165^\circ \sin 15^\circ$ by changing the product into a sum. [2]

$2\sin A\sin B = \cos(A-B)-\cos(A+B)$

$4\sin 165^\circ\sin 15^\circ = 2\left(\cos 150^\circ - \cos 180^\circ\right)$

$= 2\left(-\frac{\sqrt{3}}{2}+1\right) = 2-\sqrt{3}$

Ans: $2-\sqrt{3}$

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3. Write $\cos 3\theta - \cos 7\theta$ in the form of a product. [1]

$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

$\cos 3\theta - \cos 7\theta = 2\sin 5\theta\sin 2\theta$

Ans: $2\sin 5\theta\sin 2\theta$

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4. Simplify $\sin 41^\circ + \sin 19^\circ$. [1]

The half sum of the angles is 30°, a standard angle.

$\sin 41^\circ + \sin 19^\circ = 2\sin 30^\circ\cos 11^\circ = \cos 11^\circ$

Ans: $\cos 11^\circ$

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5. Evaluate $\cos 135^\circ + \cos 105^\circ$ using a sum to product formula. [2]

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos 135^\circ + \cos 105^\circ = 2\cos 120^\circ\cos 15^\circ$

$= \frac{-\sqrt{6}-\sqrt{2}}{4}$

Ans: $\frac{-\sqrt{6}-\sqrt{2}}{4}$

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6. Show that $\cos 49^\circ + \cos 71^\circ + \cos 169^\circ = 0$. [3]

$\cos 169^\circ + \cos 71^\circ = 2\cos\frac{169^\circ+71^\circ}{2}\cos\frac{169^\circ-71^\circ}{2} = 2\cos 120^\circ\cos 49^\circ$

$= 2\left(-\frac{1}{2}\right)\cos 49^\circ = -\cos 49^\circ$

$\therefore \cos 49^\circ + \cos 71^\circ + \cos 169^\circ = \cos 49^\circ - \cos 49^\circ = 0$

Ans: Shown: the expression is 0

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7. Evaluate $\frac{\sin 82^\circ - \sin 8^\circ}{\cos 8^\circ - \cos 82^\circ}$. [1]

$\frac{\sin 82^\circ-\sin 8^\circ}{\cos 8^\circ-\cos 82^\circ} = \frac{2\cos 45^\circ\sin 37^\circ}{2\sin 45^\circ\sin 37^\circ}$

$= \cot 45^\circ = 1$

Ans: 1

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8. Show that $\frac{\sin 8\theta - \sin 4\theta}{\cos 8\theta + \cos 4\theta} = \tan 2\theta$. [2]

$\frac{\sin 8\theta - \sin 4\theta}{\cos 8\theta + \cos 4\theta} = \frac{2\cos 6\theta \sin 2\theta}{2\cos 6\theta \cos 2\theta}$

$= \tan 2\theta$

Ans: $\tan 2\theta$

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9. Prove that $\frac{\sin 7\theta + \sin 6\theta + \sin 8\theta}{\cos 7\theta + \cos 6\theta + \cos 8\theta} = \tan 7\theta$. [3]

$\sin 8\theta + \sin 6\theta = 2\sin 7\theta \cos \theta$, $\cos 8\theta + \cos 6\theta = 2\cos 7\theta \cos \theta$

$\text{Numerator} = \sin 7\theta(2\cos\theta + 1)$, $\text{Denominator} = \cos 7\theta(2\cos\theta + 1)$

Cancel the common factor $2\cos\theta + 1$: the fraction is $\tan 7\theta$.

Ans: $\tan 7\theta$

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10. Show that $\tan 10^\circ \tan 50^\circ \tan 70^\circ = \frac{\sqrt{3}}{3}$. [5]

By the product to sum formulas, $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{4}\sin 30^\circ$.

($\sin 50^\circ \sin 70^\circ = \frac{1}{2}\cos 20^\circ + \frac{1}{4}$, then multiply by $\sin 10^\circ$.)

In the same way $\cos 10^\circ \cos 50^\circ \cos 70^\circ = \frac{1}{4}\cos 30^\circ$.

$\therefore \tan 10^\circ \tan 50^\circ \tan 70^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \tan 30^\circ = \frac{\sqrt{3}}{3}$

Ans: Shown: $\frac{\sqrt{3}}{3}$

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11. In a triangle ABC, express $\sin(2B + 2C)$ in terms of A. [1]

In a triangle, $B + C = 180^\circ - A$.

$\sin(2B + 2C) = -\sin 2A$

Ans: $-\sin 2A$

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12. Verify $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$ for the triangle with $A = 120^\circ$, $B = 30^\circ$, $C = 30^\circ$. [2]

LHS $= \frac{1}{2}$

RHS $= \frac{1}{2}$

Both sides equal $\frac{1}{2}$, so the identity holds.

Ans: Both sides equal $\frac{1}{2}$

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13. In a triangle ABC, prove that $-\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \sin B \sin C$. [5]

$A + B + C = 180^\circ$, so $A + B = 180^\circ - C$ and similar.

$\cos^2 B + \cos^2 C = 1 + \frac{1}{2}(\cos 2B + \cos 2C) = 1 + \cos(B + C)\cos(B - C) = 1 - \cos A \cos(B - C)$

$-\cos^2 A = -\cos A \cos A = \cos A \cos(B + C)$

$\text{LHS} = 1 - \cos A[\cos(B - C) - \cos(B + C)] = 1 - \cos A \cdot 2\sin B \sin C$

$= 1 - 2\cos A \sin B \sin C = \text{RHS}$

Ans: Proved

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14. In a triangle ABC, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. [5]

$A+B+C=180^\circ$, so $A+B=180^\circ-C$ and similar.

$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}=1-\frac{1}{2}(\cos A+\cos B)=1-\cos\frac{A+B}{2}\cos\frac{A-B}{2}=1-\sin\frac{C}{2}\cos\frac{A-B}{2}$

$\sin^2\frac{C}{2}=\sin\frac{C}{2}\cos\frac{A+B}{2}$

$\text{LHS}=1-\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]=1-\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}$

$=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\text{RHS}$

Ans: Proved

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15. If $\cos x+\cos y=\frac{1}{3}$ and $\sin x+\sin y=\frac{3}{7}$, find $\tan\frac{x+y}{2}$. [2]

Transforming: $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{1}{3}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{3}{7}$.

Dividing: $\tan\frac{x+y}{2}=\frac{9}{7}$

Ans: $\tan\frac{x+y}{2}=\frac{9}{7}$

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16. If $\cos x+\cos y=\frac{1}{6}$ and $\sin x-\sin y=\frac{1}{4}$, find $\sin(x-y)$ and $\cos(x-y)$. [3]

Transforming: $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{1}{6}$ and $2\cos\frac{x+y}{2}\sin\frac{x-y}{2}=\frac{1}{4}$.

Dividing: $\tan\frac{x-y}{2}=\frac{3}{2}$

With $t=\frac{3}{2}$: $\sin(x-y)=\frac{2t}{1+t^2}=\frac{12}{13}$, $\cos(x-y)=\frac{1-t^2}{1+t^2}=-\frac{5}{13}$

Ans: $\sin(x-y)=\frac{12}{13}$, $\cos(x-y)=-\frac{5}{13}$

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17. If $\cos x-\cos y=\frac{1}{2}$ and $\sin x+\sin y=\frac{2}{3}$, find $\tan\frac{x-y}{2}$, $\sin(x-y)$, $\cos(x-y)$ and $\tan(x-y)$. [5]

Transforming: $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}=\frac{1}{2}$ and $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{2}{3}$.

Dividing: $\tan\frac{x-y}{2}=-\frac{3}{4}$

With $t=-\frac{3}{4}$: $\sin(x-y)=\frac{2t}{1+t^2}=-\frac{24}{25}$, $\cos(x-y)=\frac{1-t^2}{1+t^2}=\frac{7}{25}$

$\tan(x-y)=\frac{\sin(x-y)}{\cos(x-y)}=-\frac{24}{7}$

Ans: $\tan\frac{x-y}{2}=-\frac{3}{4}$, $\sin(x-y)=-\frac{24}{25}$, $\cos(x-y)=\frac{7}{25}$, $\tan(x-y)=-\frac{24}{7}$

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18. If $\cos x+\cos y=\frac{4}{3}$ and $\sin x-\sin y=\frac{2}{7}$, find $\cos(x+y)$. [2]

Square both and add, using $\cos^2+\sin^2=1$:

$a^2+b^2=2+2(\cos x\cos y-\sin x\sin y)=2+2\cos(x+y)$

$\frac{16}{9}+\frac{4}{49}=2+2\cos(x+y)$

$\cos(x+y)=-\frac{31}{441}$

Ans: $\cos(x+y)=-\frac{31}{441}$

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19. If $\cos x+\cos y=\frac{4}{3}$ and $\sin x+\sin y=\frac{2}{3}$, find $\cos(x-y)$ and $\cos\frac{x-y}{2}$. [3]

Square both and add, using $\cos^2+\sin^2=1$:

$a^2+b^2=2+2(\cos x\cos y+\sin x\sin y)=2+2\cos(x-y)$

$\frac{16}{9}+\frac{4}{9}=2+2\cos(x-y)$

$\cos(x-y)=\frac{1}{9}$

$\cos^2\frac{x-y}{2}=\frac{1+\cos(x-y)}{2}=\frac{5}{9}$

$\cos\frac{x-y}{2}=\pm\frac{\sqrt{5}}{3}$

Ans: $\cos(x-y)=\frac{1}{9}$, $\cos\frac{x-y}{2}=\pm\frac{\sqrt{5}}{3}$

20. In a triangle ABC, $A = 30^\circ$ and $B = 45^\circ$. Find $\cos A + \cos B + \cos C$ and check that it does not exceed $\frac{3}{2}$. [1]

$C = 180^\circ - 30^\circ - 45^\circ = 105^\circ$

$\cos 30^\circ + \cos 45^\circ + \cos 105^\circ = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{-\sqrt{6}+\sqrt{2}}{4} = \frac{-\sqrt{6}+2\sqrt{3}+3\sqrt{2}}{4}$

$\frac{-\sqrt{6}+2\sqrt{3}+3\sqrt{2}}{4} \approx 1.314 \leq 1.5$

Ans: $\frac{-\sqrt{6}+2\sqrt{3}+3\sqrt{2}}{4}$
