EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Show that the constant function $f(x) = 11$ is periodic, and say whether it has a fundamental period. [1 mark]

2. Write down the periods of $\operatorname{cosec} x$ and $|\sin x|$. [1 mark]

3. Find the period of $\tan(-3x + 2)$. [1 mark]

4. Find the period of $2\cos\left(\frac{\pi x}{4} + \frac{\pi}{6}\right) + 1$. [2 marks]

5. Find the period of $\cot\frac{x}{4} - \cos\frac{2x}{3}$. [2 marks]

6. Find the period of $f(x) = \cot\frac{x}{2} + \tan\frac{x}{3} + \tan\frac{3x}{4}$. [5 marks]

7. Find the period of $\cos^3 x$. [1 mark]

8. Find the period of $\operatorname{cosec}^4 2x$. [2 marks]

9. Find the period of $6\sin^5 x + 3\cos^5 x$. [2 marks]

10. Find the period of $4|\operatorname{cosec} 4x| + 4|\sec 4x|$. [3 marks]

11. Find the period of $\frac{x}{3} - \left[\frac{x}{3}\right]$. [1 mark]

12. Find the period of $f(x) = 2x - [2x] + \tan\frac{\pi x}{2}$. [3 marks]

13. Write the range of $\sin^2 x$. [1 mark]

14. Find the range of $4 + 5|\sin x|$. [2 marks]

15. Find the maximum and minimum values of $15\cos x + 8\sin x + 1$. [2 marks]

16. Find the maximum and minimum values of $f(x) = -6\sin\left(x + \frac{\pi}{6}\right) + 16\cos x + 5$. [5 marks]

17. Find the range of $\dfrac{1}{-2 - \cos x}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{15\cos 6x + 8\sin 6x + 18}$. [5 marks]

19. Find the minimum value of $49\tan\theta + 25\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $25\sin^2\theta + 64\,\mathrm{cosec}^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	Periodic; no fundamental period
2	2π and π
3	$\frac{\pi}{3}$
4	8
5	12π
6	12π
7	2π
8	$\frac{\pi}{2}$
9	2π
10	$\frac{\pi}{8}$
11	3
12	2
13	$[0, 1]$
14	$[4, 9]$
15	Maximum 18, minimum -16
16	Maximum 19, minimum -9
17	$[-1, -\frac{1}{3}]$
18	Period $\frac{\pi}{3}$; range $[\frac{1}{35}, 1]$
19	70
20	Minimum 89, at $\sin^2\theta = 1$

Solutions

1. Show that the constant function $f(x) = 11$ is periodic, and say whether it has a fundamental period. [1]

$f(x+p) = 11 = f(x)$ for every positive number p.
So f is periodic.
There is no least positive number, so f has no fundamental period.

Ans: Periodic; no fundamental period

---

2. Write down the periods of $\operatorname{cosec} x$ and $|\sin x|$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.
So the periods are 2π and π.

Ans: 2π and π

---

3. Find the period of $\tan(-3x+2)$. [1]

The period of $\tan x$ is π; for $f(ax+b)$ the period is $\frac{p}{|a|}$.
$\text{Period} = \frac{\pi}{3}$

Ans: $\frac{\pi}{3}$

---

4. Find the period of $2\cos\left(\frac{\pi x}{4} + \frac{\pi}{6}\right) + 1$. [2]

The period of $\cos x$ is 2π, and here $a = \frac{\pi}{4}$.
The constants 2 and 1 do not change the period.
$\text{Period} = \frac{2\pi}{\frac{\pi}{4}} = 8$

Ans: 8

---

5. Find the period of $\cot\frac{x}{4} - \cos\frac{2x}{3}$. [2]

$\text{Period of } \cot\frac{x}{4} = 4\pi, \quad \text{period of } \cos\frac{2x}{3} = 3\pi$
$\text{L.C.M.}\{4\pi, 3\pi\} = 12\pi$
Period $= 12\pi$.

Ans: 12π

---

6. Find the period of $f(x) = \cot\frac{x}{2} + \tan\frac{x}{3} + \tan\frac{3x}{4}$. [5]

$\text{Period of } \cot\frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$
$\text{Period of } \tan\frac{x}{3} = \frac{\pi}{\frac{1}{3}} = 3\pi$
$\text{Period of } \tan\frac{3x}{4} = \frac{\pi}{\frac{3}{4}} = \frac{4\pi}{3}$
Write each period as a fraction times π: $\frac{2}{1}\pi$, $\frac{3}{1}\pi$, $\frac{4}{3}\pi$.
$\text{L.C.M.} = \frac{\text{L.C.M.}\{2,3,4\}}{\text{G.C.D.}\{1,1,3\}}\,\pi = \frac{12}{1}\,\pi$
No two terms swap into each other, so the period of f is the L.C.M.: 12π.

Ans: 12π

---

7. Find the period of $\cos^3 x$. [1]

$\cos^n x$ has period π when n is even and 2π when n is odd; here $n = 3$ is odd.

Period $= 2\pi$.

Ans: 2π

---

8. Find the period of $\operatorname{cosec}^4 2x$. [2]

$\operatorname{cosec}^4 x$ has period π (4 is even).

$\text{Period of } \operatorname{cosec}^4 2x = \frac{\pi}{2}$

Ans: $\frac{\pi}{2}$

---

9. Find the period of $6\sin^5 x + 3\cos^5 x$. [2]

$n = 5$ is odd.

From the table for $a\sin^n x + b\cos^n x$, the period is 2π.

Ans: 2π

---

10. Find the period of $4|\operatorname{cosec} 4x| + 4|\sec 4x|$. [3]

$a|\operatorname{cosec} x| + b|\sec x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are equal, so with argument x the period would be $\frac{\pi}{2}$.

The argument is $4x$, so divide by 4:

$\text{Period} = \frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$

Ans: $\frac{\pi}{8}$

---

11. Find the period of $\frac{x}{3} - \left[\frac{x}{3}\right]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{\frac{1}{3}} = 3$

Ans: 3

---

12. Find the period of $f(x) = 2x - [2x] + \tan\frac{\pi x}{2}$. [3]

$\text{Period of } 2x - [2x] = \frac{1}{2}$

$\text{Period of } \tan\frac{\pi x}{2} = \frac{\pi}{\frac{\pi}{2}} = 2$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{2}, 2\right\} = \frac{\text{L.C.M.}\{1,2\}}{\text{G.C.D.}\{2,1\}} = 2$

Ans: 2

---

13. Write the range of $\sin^2 x$. [1]

From the table of ranges, $\sin^2 x$ takes the values $[0, 1]$.

Ans: $[0, 1]$

---

14. Find the range of $4 + 5|\sin x|$. [2]

The range of $|\sin x|$ is $[0, 1]$.

$0 \le |\sin x| \le 1$

Multiply by 5 and add 4.

Range $= [4, 9]$.

Ans: $[4, 9]$

---

15. Find the maximum and minimum values of $15\cos x + 8\sin x + 1$. [2]

$a = 15,\ b = 8,\ c = 1$

$\sqrt{a^2 + b^2} = \sqrt{225 + 64} = 17$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 18, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -16$

Ans: Maximum 18**, minimum** -16

---

16. Find the maximum and minimum values of $f(x) = -6\sin\left(x + \frac{\pi}{6}\right) + 16\cos x + 5$. [5]

Expand with $\sin(x \pm \beta) = \sin x \cos\beta \pm \cos x \sin\beta$:

$f(x) = -3\sqrt{3}\sin x - 3\cos x + 16\cos x + 5$

$= (13)\cos x + \left(-3\sqrt{3}\right)\sin x + 5$

$a^2 + b^2 = 169 + 27 = 196$, so $\sqrt{a^2 + b^2} = 14$

$\text{Maximum} = 5 + 14 = 19, \quad \text{minimum} = 5 - 14 = -9$

Ans: Maximum 19**, minimum** -9

---

17. Find the range of $\dfrac{1}{-2 - \cos x}$. [3]

$-1 \le \cos x \le 1 \Rightarrow -3 \le -2 - \cos x \le -1$

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range $= [-1, -\frac{1}{3}]$.

Ans: $[-1, -\frac{1}{3}]$

---

18. Find the period and the range of $f(x) = \dfrac{1}{15\cos 6x + 8\sin 6x + 18}$. [5]

The denominator $g(x) = 15\cos 6x + 8\sin 6x + 18$ has period $\frac{2\pi}{6} = \frac{\pi}{3}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{225 + 64} = 17$

Range of g: $[18 - 17, 18 + 17] = [1, 35]$.

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range of f: $[\frac{1}{35}, 1]$.

Ans: Period $\frac{\pi}{3}$**; range** $[\frac{1}{35}, 1]$

---

19. Find the minimum value of $49\tan\theta + 25\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$49\tan\theta + 25\cot\theta \ge 2\sqrt{49\tan\theta \cdot 25\cot\theta} = 2\sqrt{1225}$

since $\tan\theta\cot\theta = 1$. Equality holds when $49\tan\theta = 25\cot\theta$, which is possible.

Minimum $= 2\sqrt{1225} = 70$.

Ans: 70

---

20. Using A.M. $\geq$ G.M., find the minimum value of $25\sin^2\theta + 64\,\mathrm{cosec}^2\theta$, and say where it is reached. [5]

$25\sin^2\theta + 64\,\mathrm{cosec}^2\theta = (5)^2\sin^2\theta + (8)^2\,\mathrm{cosec}^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $25\sin^2\theta + 64\,\mathrm{cosec}^2\theta \geq 2\sqrt{25 \cdot 64} = 80$

Equality needs $25\sin^2\theta = 64\,\mathrm{cosec}^2\theta$, that is $\sin^2\theta = \frac{8}{5}$.

But $\frac{8}{5} > 1$, so equality is impossible; the bound 80 is not reached.

With $u = \sin^2\theta \in (0, 1]$, the value $25u + \frac{64}{u}$ decreases as u increases up to 1.

Minimum at $\sin^2\theta = 1$: $25 + 64 = 89$.

Ans: Minimum 89**, at** $\sin^2\theta = 1$