EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 08

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. The fundamental period of $f(x)$ is $\frac{2\pi}{9}$. What is the period of $f(x) + 4$? [1 mark]

2. Write down the periods of $|\cot x|$ and $\sec x$. [1 mark]

3. Find the period of $\cos(5x - 2)$. [1 mark]

4. Find the period of $2\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) + 1$. [2 marks]

5. Find the period of $\tan\frac{3x}{2} + \sin\frac{x}{3}$. [2 marks]

6. Find the period of $f(x) = \sec\frac{2x}{5} - \operatorname{cosec}\frac{5x}{4} + \cos\frac{x}{2}$. [5 marks]

7. Find the period of $\cot^2 x$. [1 mark]

8. Find the period of $\tan^2\frac{2x}{3}$. [2 marks]

9. Find the period of $3\tan^4 x + 3\cot^4 x$. [2 marks]

10. Find the period of $4|\sin\frac{x}{3}| + |\cos\frac{x}{3}|$. [3 marks]

11. Find the period of $6x - [6x]$. [1 mark]

12. Find the period of $f(x) = \frac{2x}{3} - [\frac{2x}{3}] + \cos\pi x$. [3 marks]

13. Write the range of $\cos^5 x$. [1 mark]

14. Find the range of $7 + 4\sin^3 x$. [2 marks]

15. Find the maximum and minimum values of $8\sin x + 6\cos x + 5$. [2 marks]

16. Find the maximum and minimum values of $f(x) = -3\cos\left(x - \frac{\pi}{6}\right) + 8\sin x + 1$. [5 marks]

17. Find the range of $\dfrac{1}{-7\cos x + 24\sin x - 25}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{15\cos 5x - 8\sin 5x}$. [5 marks]

19. Find the minimum value of $1\cot\theta + 49\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $1\cos^2\theta + 49\sec^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	$\frac{2\pi}{9}$
2	π and 2π
3	$\frac{2\pi}{5}$
4	4π
5	6π
6	40π
7	π
8	$\frac{3\pi}{2}$
9	$\frac{\pi}{2}$
10	3π
11	$\frac{1}{6}$
12	6
13	$[-1, 1]$
14	$[3, 11]$
15	Maximum 15, minimum -5
16	Maximum 8, minimum -6
17	$(-\infty, -\frac{1}{50}]$
18	Period $\frac{2\pi}{5}$; range $(-\infty, -\frac{1}{17}] \cup [\frac{1}{17}, \infty)$
19	14
20	Minimum 50, at $\cos^2\theta = 1$

Solutions

1. The fundamental period of $f(x)$ is $\frac{2\pi}{9}$. What is the period of $f(x)+4$? [1]

If f has period p, then $-f$ and $\frac{1}{f}$ have period p; multiplying by a constant or adding a constant does not change the period.

So the period of $f(x)+4$ is $\frac{2\pi}{9}$.

Ans: $\frac{2\pi}{9}$

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2. Write down the periods of $|\cot x|$ and $\sec x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are π and 2π.

Ans: π **and** 2π

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3. Find the period of $\cos(5x-2)$. [1]

The period of $\cos x$ is 2π; for $f(ax+b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{2\pi}{5}$

Ans: $\frac{2\pi}{5}$

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4. Find the period of $2\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)+1$. [2]

The period of $\cos x$ is 2π, and here $a=\frac{1}{2}$.

The constants 2 and 1 do not change the period.

$\text{Period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Ans: 4π

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5. Find the period of $\tan\frac{3x}{2}+\sin\frac{x}{3}$. [2]

$\text{Period of } \tan\frac{3x}{2} = \frac{2\pi}{3}, \quad \text{period of } \sin\frac{x}{3} = 6\pi$

$\text{L.C.M.}\left\{\frac{2\pi}{3}, 6\pi\right\} = 6\pi$

Period $= 6\pi$.

Ans: 6π

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6. Find the period of $f(x) = \sec\frac{2x}{5} - \operatorname{cosec}\frac{5x}{4} + \cos\frac{x}{2}$. [5]

$\text{Period of } \sec\frac{2x}{5} = \frac{2\pi}{\frac{2}{5}} = 5\pi$

$\text{Period of } \operatorname{cosec}\frac{5x}{4} = \frac{2\pi}{\frac{5}{4}} = \frac{8\pi}{5}$

$\text{Period of } \cos\frac{x}{2} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Write each period as a fraction times π: $\frac{5}{1}\pi$, $\frac{8}{5}\pi$, $\frac{4}{1}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{5,8,4\}}{\text{G.C.D.}\{1,5,1\}}\pi = \frac{40}{1}\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 40π.

Ans: 40π

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7. Find the period of $\cot^2 x$. [1]

$\tan^n x$ and $\cot^n x$ have period π for every n.

Period $= \pi$.

Ans: π

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8. Find the period of $\tan^2 \frac{2x}{3}$. [2]

$\tan^2 x$ has period π (tangent family).

$\text{Period of } \tan^2 \frac{2x}{3} = \frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}$

Ans: $\frac{3\pi}{2}$

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9. Find the period of $3\tan^4 x + 3\cot^4 x$. [2]

$n = 4$ is even and the coefficients are equal.

Shifting x by $\frac{\pi}{2}$ swaps the two terms, so the sum repeats after $\frac{\pi}{2}$.

From the table for $a\tan^n x + b\cot^n x$, the period is $\frac{\pi}{2}$.

Ans: $\frac{\pi}{2}$

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10. Find the period of $4|\sin \frac{x}{3}| + |\cos \frac{x}{3}|$. [3]

$a|\sin x| + b|\cos x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are unequal, so with argument x the period would be π.

The argument is $\frac{x}{3}$, so divide by $\frac{1}{3}$:

$\text{Period} = \frac{\pi}{\frac{1}{3}} = 3\pi$

Ans: 3π

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11. Find the period of $6x - [6x]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{6}$

Ans: $\frac{1}{6}$

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12. Find the period of $f(x) = \frac{2x}{3} - [\frac{2x}{3}] + \cos \pi x$. [3]

$\text{Period of } \frac{2x}{3} - [\frac{2x}{3}] = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

$\text{Period of } \cos \pi x = \frac{2\pi}{\pi} = 2$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.} \left\{\frac{3}{2}, 2\right\} = \frac{\text{L.C.M.}\{3,2\}}{\text{G.C.D.}\{2,1\}} = 6$

Ans: 6

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13. Write the range of $\cos^5 x$. [1]

From the table of ranges, $\cos^5 x$ takes the values $[-1, 1]$.

Ans: $[-1, 1]$

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14. Find the range of $7 + 4\sin^3 x$. [2]

The range of $\sin^3 x$ is $[-1, 1]$.

$-1 \le \sin^3 x \le 1$

Multiply by 4 and add 7.

Range $= [3, 11]$.

Ans: $[3, 11]$

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15. Find the maximum and minimum values of $8\sin x + 6\cos x + 5$. [2]

$a = 6,\ b = 8,\ c = 5$

$\sqrt{a^2 + b^2} = \sqrt{36 + 64} = 10$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 15, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -5$

Ans: Maximum 15**, minimum** -5

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16. Find the maximum and minimum values of $f(x) = -3\cos\left(x - \frac{\pi}{6}\right) + 8\sin x + 1$. [5]

Expand with $\cos(x \pm \beta) = \cos x \cos\beta \mp \sin x \sin\beta$:

$f(x) = -\frac{3}{2}\sin x - \frac{3\sqrt{3}}{2}\cos x + 8\sin x + 1$

$= \left(-\frac{3\sqrt{3}}{2}\right)\cos x + \left(\frac{13}{2}\right)\sin x + 1$

$a^2 + b^2 = \frac{27}{4} + \frac{169}{4} = 49$, so $\sqrt{a^2 + b^2} = 7$

$\text{Maximum} = 1 + 7 = 8, \quad \text{minimum} = 1 - 7 = -6$

Ans: Maximum 8**, minimum** -6

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17. Find the range of $\dfrac{1}{-7\cos x + 24\sin x - 25}$. [3]

$\sqrt{49 + 576} = 25$

The denominator has range $[-50, 0]$.

The range of f is $[a, 0]$, so the range of $\frac{1}{f}$ (where defined) is $\left(-\infty, \frac{1}{a}\right]$.

Range $= (-\infty, -\frac{1}{50}]$.

Ans: $(-\infty, -\frac{1}{50}]$

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18. Find the period and the range of $f(x) = \dfrac{1}{15\cos 5x - 8\sin 5x}$. [5]

The denominator $g(x) = 15\cos 5x - 8\sin 5x$ has period $\frac{2\pi}{5} = \frac{2\pi}{5}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{225 + 64} = 17$

Range of g: $[0 - 17, 0 + 17] = [-17, 17]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range of f: $(-\infty, -\frac{1}{17}] \cup [\frac{1}{17}, \infty)$.

Ans: Period $\frac{2\pi}{5}$**; range** $(-\infty, -\frac{1}{17}] \cup [\frac{1}{17}, \infty)$

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19. Find the minimum value of $1\cot\theta + 49\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$1\cot\theta + 49\tan\theta \ge 2\sqrt{1\cot\theta \cdot 49\tan\theta} = 2\sqrt{49}$

since $\tan\theta\cot\theta = 1$. Equality holds when $1\cot\theta = 49\tan\theta$, which is possible.

Minimum $= 2\sqrt{49} = 14$.

Ans: 14

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20. Using A.M. $\geq$ G.M., find the minimum value of $1\cos^2\theta + 49\sec^2\theta$, and say where it is reached. [5]

$1\cos^2\theta + 49\sec^2\theta = (1)^2\cos^2\theta + (7)^2\sec^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $1\cos^2\theta + 49\sec^2\theta \geq 2\sqrt{1 \cdot 49} = 14$

Equality needs $1\cos^2\theta = 49\sec^2\theta$, that is $\cos^2\theta = 7$.

But $7 > 1$, so equality is impossible; the bound 14 is not reached.

With $u = \cos^2\theta \in (0, 1]$, the value $1u + \frac{49}{u}$ decreases as u increases up to 1.

Minimum at $\cos^2\theta = 1$: $1 + 49 = 50$.

Ans: Minimum 50, at $\cos^2\theta = 1$