EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. f is a periodic function with period 6, and $f(1) = -2$. Find $f(-17)$. [1 mark]

2. Write down the periods of $\sec x$ and $\tan x$. [1 mark]

3. Find the period of $\cot 5x$. [1 mark]

4. Find the period of $2\sec\left(\frac{2x}{3} + 7\right) + 3$. [2 marks]

5. Find the period of $\operatorname{cosec} \frac{2x}{3} - \sec \frac{x}{4}$. [2 marks]

6. Find the period of $f(x) = \cot \frac{5x}{3} + \cot \frac{x}{4} + \sin \frac{2x}{5}$. [5 marks]

7. Find the period of $\tan^7 x$. [1 mark]

8. Find the period of $\sin^2 \frac{x}{2}$. [2 marks]

9. Find the period of $5\operatorname{cosec} x + 5\sec x$. [2 marks]

10. Find the period of $3|\operatorname{cosec} \frac{x}{3}| + 3|\sec \frac{x}{3}|$. [3 marks]

11. Find the period of $8x - [8x]$. [1 mark]

12. Find the period of $f(x) = 4x - [4x] + \cot \frac{2\pi x}{3}$. [3 marks]

13. What is the range of $\sec^2 x$? [1 mark]

14. Find the range of $-1 + 3\operatorname{cosec}^2 x$. [2 marks]

15. Find the maximum and minimum values of $2\cos x + 3\sin x + 5$. [2 marks]

16. Find the maximum and minimum values of $f(x) = 5\sin\left(x - \frac{\pi}{3}\right) + 16\sin x + 3$. [5 marks]

17. Find the range of $\dfrac{1}{-3 - \sin x}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{7\cos 3x - 24\sin 3x - 2}$. [5 marks]

19. Find the minimum value of $2\tan\theta + 18\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $4\sin^2\theta + 49\,\mathrm{cosec}^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	-2
2	2π and π
3	$\frac{\pi}{5}$
4	3π
5	24π
6	60π
7	π
8	2π
9	2π
10	$\frac{3\pi}{2}$
11	$\frac{1}{8}$
12	$\frac{3}{2}$
13	$[1, \infty)$
14	$[2, \infty)$
15	Maximum $\sqrt{13}+5$, minimum $5-\sqrt{13}$
16	Maximum 22, minimum -16
17	$[-\frac{1}{2}, -\frac{1}{4}]$
18	Period $\frac{2\pi}{3}$; range $(-\infty, -\frac{1}{27}] \cup [\frac{1}{23}, \infty)$
19	12
20	Minimum 53, at $\sin^2\theta = 1$

Solutions

1. f is a periodic function with period 6, and $f(1) = -2$. Find $f(-17)$. [1]

$-17 = 1 - 3 \times 6$

$f(x + np) = f(x) \Rightarrow f(-17) = f(1) = -2$

Ans: -2

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2. Write down the periods of $\sec x$ and $\tan x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are 2π and π.

Ans: 2π **and** π

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3. Find the period of $\cot 5x$. [1]

The period of $\cot x$ is π; for $f(ax + b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{\pi}{5}$

Ans: $\frac{\pi}{5}$

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4. Find the period of $2\sec\left(\frac{2x}{3} + 7\right) + 3$. [2]

The period of $\sec x$ is 2π, and here $a = \frac{2}{3}$.

The constants 2 and 3 do not change the period.

$\text{Period} = \frac{2\pi}{\frac{2}{3}} = 3\pi$

Ans: 3π

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5. Find the period of $\operatorname{cosec} \frac{2x}{3} - \sec \frac{x}{4}$. [2]

$\text{Period of } \operatorname{cosec} \frac{2x}{3} = 3\pi, \quad \text{period of } \sec \frac{x}{4} = 8\pi$

$\text{L.C.M.}\{3\pi, 8\pi\} = 24\pi$

Period $= 24\pi$.

Ans: 24π

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6. Find the period of $f(x) = \cot \frac{5x}{3} + \cot \frac{x}{4} + \sin \frac{2x}{5}$. [5]

$\text{Period of } \cot \frac{5x}{3} = \frac{\pi}{\frac{5}{3}} = \frac{3\pi}{5}$

$\text{Period of } \cot \frac{x}{4} = \frac{\pi}{\frac{1}{4}} = 4\pi$

$\text{Period of } \sin \frac{2x}{5} = \frac{2\pi}{\frac{2}{5}} = 5\pi$

Write each period as a fraction times π: $\frac{3}{5}\pi$, $\frac{4}{1}\pi$, $\frac{5}{1}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{3,4,5\}}{\text{G.C.D.}\{5,1,1\}}\,\pi = \frac{60}{1}\,\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 60π.

Ans: 60π

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7. Find the period of $\tan^7 x$. [1]

$\tan^n x$ and $\cot^n x$ have period π for every n.

Period $= \pi$.

Ans: π

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8. Find the period of $\sin^2 \frac{x}{2}$. [2]

$\sin^2 x$ has period π (2 is even).

$\text{Period of } \sin^2 \frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$

Ans: 2π

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9. Find the period of $5\operatorname{cosec} x + 5\sec x$. [2]

$n = 1$ is odd.

From the table for $a\operatorname{cosec}^n x + b\sec^n x$, the period is 2π.

Ans: 2π

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10. Find the period of $3|\operatorname{cosec} \frac{x}{3}| + 3|\sec \frac{x}{3}|$. [3]

$a|\operatorname{cosec} x| + b|\sec x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are equal, so with argument x the period would be $\frac{\pi}{2}$.

The argument is $\frac{x}{3}$, so divide by $\frac{1}{3}$:

$\text{Period} = \frac{\frac{\pi}{2}}{\frac{1}{3}} = \frac{3\pi}{2}$

Ans: $\frac{3\pi}{2}$

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11. Find the period of $8x - [8x]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{8}$

Ans: $\frac{1}{8}$

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12. Find the period of $f(x) = 4x - [4x] + \cot \frac{2\pi x}{3}$. [3]

$\text{Period of } 4x - [4x] = \frac{1}{4}$

$\text{Period of } \cot \frac{2\pi x}{3} = \frac{\pi}{\frac{2\pi}{3}} = \frac{3}{2}$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.} \left\{\frac{1}{4}, \frac{3}{2}\right\} = \frac{\text{L.C.M.}\{1,3\}}{\text{G.C.D.}\{4,2\}} = \frac{3}{2}$

Ans: $\frac{3}{2}$

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13. What is the range of $\sec^2 x$? [1]

From the table of ranges, $\sec^2 x$ takes the values $[1, \infty)$.

Ans: $[1, \infty)$

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14. Find the range of $-1 + 3\,\mathrm{cosec}^2\,x$. [2]

The range of $\mathrm{cosec}^2\,x$ is $[1, \infty)$.

$1 \le \mathrm{cosec}^2\,x < \infty$

Multiply by 3 and add -1.

Range $= [2, \infty)$.

Ans: $[2, \infty)$

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15. Find the maximum and minimum values of $2\cos x + 3\sin x + 5$. [2]

$a = 2,\ b = 3,\ c = 5$

$\sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13}$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = \sqrt{13} + 5, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = 5 - \sqrt{13}$

Ans: Maximum $\sqrt{13} + 5$**, minimum** $5 - \sqrt{13}$

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16. Find the maximum and minimum values of $f(x) = 5\sin\left(x - \frac{\pi}{3}\right) + 16\sin x + 3$. [5]

Expand with $\sin(x \pm \beta) = \sin x \cos\beta \pm \cos x \sin\beta$:

$f(x) = \frac{5}{2}\sin x - \frac{5\sqrt{3}}{2}\cos x + 16\sin x + 3$

$= \left(-\frac{5\sqrt{3}}{2}\right)\cos x + \left(\frac{37}{2}\right)\sin x + 3$

$a^2 + b^2 = \frac{75}{4} + \frac{1369}{4} = 361$, so $\sqrt{a^2 + b^2} = 19$

$\text{Maximum} = 3 + 19 = 22, \quad \text{minimum} = 3 - 19 = -16$

Ans: Maximum 22**, minimum** -16

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17. Find the range of $\dfrac{1}{-3 - \sin x}$. [3]

$-1 \le \sin x \le 1 \Rightarrow -4 \le -3 - \sin x \le -2$

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range $= [-\frac{1}{2}, -\frac{1}{4}]$.

Ans: $[-\frac{1}{2}, -\frac{1}{4}]$

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18. Find the period and the range of $f(x) = \dfrac{1}{7\cos 3x - 24\sin 3x - 2}$. [5]

The denominator $g(x) = 7\cos 3x - 24\sin 3x - 2$ has period $\frac{2\pi}{3} = \frac{2\pi}{3}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{49 + 576} = 25$

Range of g: $[-2 - 25, -2 + 25] = [-27, 23]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range of f: $(-\infty, -\frac{1}{27}] \cup [\frac{1}{23}, \infty)$.

Ans: Period $\frac{2\pi}{3}$**; range** $(-\infty, -\frac{1}{27}] \cup [\frac{1}{23}, \infty)$

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19. Find the minimum value of $2\tan\theta + 18\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$2\tan\theta + 18\cot\theta \ge 2\sqrt{2\tan\theta \cdot 18\cot\theta} = 2\sqrt{36}$

since $\tan\theta\cot\theta = 1$. Equality holds when $2\tan\theta = 18\cot\theta$, which is possible.

Minimum $= 2\sqrt{36} = 12$.

Ans: 12

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20. Using A.M. $\geq$ G.M., find the minimum value of $4\sin^2\theta + 49\,\mathrm{cosec}^2\theta$, and say where it is reached. [5]

$4\sin^2\theta + 49\,\mathrm{cosec}^2\theta = (2)^2\sin^2\theta + (7)^2\,\mathrm{cosec}^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $4\sin^2\theta + 49\,\mathrm{cosec}^2\theta \geq 2\sqrt{4 \cdot 49} = 28$

Equality needs $4\sin^2\theta = 49\,\mathrm{cosec}^2\theta$, that is $\sin^2\theta = \frac{7}{2}$.

But $\frac{7}{2} > 1$, so equality is impossible; the bound 28 is not reached.

With $u = \sin^2\theta \in (0, 1]$, the value $4u + \frac{49}{u}$ decreases as u increases up to 1.

Minimum at $\sin^2\theta = 1$: $4 + 49 = 53$.

Ans: Minimum 53, at $\sin^2\theta = 1$
