EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 06

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Show that the constant function $f(x) = 13$ is periodic, and say whether it has a fundamental period. [1 mark]

2. Write down the periods of $|\sin x|$ and $\cot x$. [1 mark]

3. Find the period of $\tan(-2x + 3)$. [1 mark]

4. Find the period of $-2\cot\left(\frac{\pi x}{2} - 2\right) + 3$. [2 marks]

5. Find the period of $\operatorname{cosec}\frac{x}{2} - \sec\frac{3x}{2}$. [2 marks]

6. Find the period of $f(x) = \sec 2x + \cos\frac{x}{4} + \cot\frac{4x}{3}$. [5 marks]

7. Find the period of $\cot^7 x$. [1 mark]

8. Find the period of $\sec^6\frac{x}{3}$. [2 marks]

9. Find the period of $5\operatorname{cosec}^2 x + 7\sec^2 x$. [2 marks]

10. Find the period of $|\operatorname{cosec} 6x| + 6|\sec 6x|$. [3 marks]

11. Find the period of $\frac{x}{5} - \left[\frac{x}{5}\right]$. [1 mark]

12. Find the period of $f(x) = 6x - [6x] + \cot \pi x$. [3 marks]

13. What is the range of $\operatorname{cosec} x$? [1 mark]

14. Find the range of $4 + 2\cos^2 x$. [2 marks]

15. Find the maximum and minimum values of $8\sin x + 6\cos x + 7$. [2 marks]

16. Find the maximum and minimum values of $f(x) = -6\cos\left(x + \frac{\pi}{6}\right) + 10\sin x + 3$. [5 marks]

17. Find the range of $\dfrac{1}{-15\cos x - 8\sin x - 20}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{-24\cos 5x - 7\sin 5x + 1}$. [5 marks]

19. Find the minimum value of $8\cot\theta + 2\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $9\cos^2\theta + 25\sec^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	Periodic; no fundamental period
2	π and π
3	$\frac{\pi}{2}$
4	2
5	4π
6	24π
7	π
8	3π
9	π
10	$\frac{\pi}{6}$
11	5
12	1
13	$(-\infty, -1] \cup [1, \infty)$
14	$[4, 6]$
15	Maximum 17, minimum -3
16	Maximum 17, minimum -11
17	$[-\frac{1}{3}, -\frac{1}{37}]$
18	Period $\frac{2\pi}{5}$; range $(-\infty, -\frac{1}{24}] \cup [\frac{1}{26}, \infty)$
19	8
20	Minimum 34, at $\cos^2\theta = 1$

Solutions

1. Show that the constant function $f(x) = 13$ is periodic, and say whether it has a fundamental period. [1]

$f(x+p) = 13 = f(x)$ for every positive number p.
So f is periodic.
There is no least positive number, so f has no fundamental period.

Ans: Periodic; no fundamental period

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2. Write down the periods of $|\sin x|$ and $\cot x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.
So the periods are π and π.

Ans: π and π

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3. Find the period of $\tan(-2x+3)$. [1]

The period of $\tan x$ is π; for $f(ax+b)$ the period is $\frac{p}{|a|}$.
$\text{Period} = \frac{\pi}{2}$

Ans: $\frac{\pi}{2}$

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4. Find the period of $-2\cot\left(\frac{\pi x}{2} - 2\right) + 3$. [2]

The period of $\cot x$ is π, and here $a = \frac{\pi}{2}$.
The constants -2 and 3 do not change the period.
$\text{Period} = \frac{\pi}{\frac{\pi}{2}} = 2$

Ans: 2

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5. Find the period of $\operatorname{cosec}\frac{x}{2} - \sec\frac{3x}{2}$. [2]

$\text{Period of } \operatorname{cosec}\frac{x}{2} = 4\pi, \quad \text{period of } \sec\frac{3x}{2} = \frac{4\pi}{3}$
$\text{L.C.M.}\left\{4\pi, \frac{4\pi}{3}\right\} = 4\pi$
Period $= 4\pi$.

Ans: 4π

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6. Find the period of $f(x) = \sec 2x + \cos\frac{x}{4} + \cot\frac{4x}{3}$. [5]

$\text{Period of } \sec 2x = \frac{2\pi}{2} = \pi$
$\text{Period of } \cos\frac{x}{4} = \frac{2\pi}{\frac{1}{4}} = 8\pi$
$\text{Period of } \cot\frac{4x}{3} = \frac{\pi}{\frac{4}{3}} = \frac{3\pi}{4}$
Write each period as a fraction times π: $\frac{1}{1}\pi$, $\frac{8}{1}\pi$, $\frac{3}{4}\pi$.
$\text{L.C.M.} = \frac{\text{L.C.M.}\{1,8,3\}}{\text{G.C.D.}\{1,1,4\}}\pi = \frac{24}{1}\pi$
No two terms swap into each other, so the period of f is the L.C.M.: 24π.

Ans: 24π

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7. Find the period of $\cot^7 x$. [1]

$\tan^n x$ and $\cot^n x$ have period π for every n.

Period $= \pi$.

Ans: π

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8. Find the period of $\sec^6 \frac{x}{3}$. [2]

$\sec^6 x$ has period π (6 is even).

$\text{Period of } \sec^6 \frac{x}{3} = \frac{\pi}{\frac{1}{3}} = 3\pi$

Ans: 3π

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9. Find the period of $5\,\text{cosec}^2 x + 7\sec^2 x$. [2]

$n = 2$ is even and the coefficients are unequal.

From the table for $a\,\text{cosec}^n x + b\sec^n x$, the period is π.

Ans: π

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10. Find the period of $|\,\text{cosec}\, 6x| + 6|\sec 6x|$. [3]

$a|\,\text{cosec}\, x| + b|\sec x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are unequal, so with argument x the period would be π.

The argument is $6x$, so divide by 6:

$\text{Period} = \frac{\pi}{6}$

Ans: $\frac{\pi}{6}$

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11. Find the period of $\frac{x}{5} - \left[\frac{x}{5}\right]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{\frac{1}{5}} = 5$

Ans: 5

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12. Find the period of $f(x) = 6x - [6x] + \cot \pi x$. [3]

$\text{Period of } 6x - [6x] = \frac{1}{6}$

$\text{Period of } \cot \pi x = \frac{\pi}{\pi} = 1$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{6}, 1\right\} = \frac{\text{L.C.M.}\{1,1\}}{\text{G.C.D.}\{6,1\}} = 1$

Ans: 1

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13. What is the range of $\text{cosec}\, x$? [1]

From the table of ranges, $\text{cosec}\, x$ takes the values $(-\infty, -1] \cup [1, \infty)$.

Ans: $(-\infty, -1] \cup [1, \infty)$

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14. Find the range of $4 + 2\cos^2 x$. [2]

The range of $\cos^2 x$ is $[0, 1]$.

$0 \le \cos^2 x \le 1$

Multiply by 2 and add 4.

Range $= [4, 6]$.

Ans: $[4, 6]$

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15. Find the maximum and minimum values of $8\sin x + 6\cos x + 7$. [2]

$a = 6,\ b = 8,\ c = 7$

$\sqrt{a^2 + b^2} = \sqrt{36 + 64} = 10$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 17, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -3$

Ans: Maximum 17**, minimum** -3

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16. Find the maximum and minimum values of $f(x) = -6\cos\left(x + \frac{\pi}{6}\right) + 10\sin x + 3$. [5]

Expand with $\cos(x \pm \beta) = \cos x \cos\beta \mp \sin x \sin\beta$:

$f(x) = 3\sin x - 3\sqrt{3}\cos x + 10\sin x + 3$

$= \left(-3\sqrt{3}\right)\cos x + (13)\sin x + 3$

$a^2 + b^2 = 27 + 169 = 196$, so $\sqrt{a^2 + b^2} = 14$

$\text{Maximum} = 3 + 14 = 17, \quad \text{minimum} = 3 - 14 = -11$

Ans: Maximum 17**, minimum** -11

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17. Find the range of $\dfrac{1}{-15\cos x - 8\sin x - 20}$. [3]

$\sqrt{225 + 64} = 17$

The denominator has range $[-37, -3]$.

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range $= [-\frac{1}{3}, -\frac{1}{37}]$.

Ans: $[-\frac{1}{3}, -\frac{1}{37}]$

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18. Find the period and the range of $f(x) = \dfrac{1}{-24\cos 5x - 7\sin 5x + 1}$. [5]

The denominator $g(x) = -24\cos 5x - 7\sin 5x + 1$ has period $\frac{2\pi}{5} = \frac{2\pi}{5}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{576 + 49} = 25$

Range of g: $[1 - 25, 1 + 25] = [-24, 26]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range of f: $(-\infty, -\frac{1}{24}] \cup [\frac{1}{26}, \infty)$.

Ans: Period $\frac{2\pi}{5}$**; range** $(-\infty, -\frac{1}{24}] \cup [\frac{1}{26}, \infty)$

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19. Find the minimum value of $8\cot\theta + 2\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$8\cot\theta + 2\tan\theta \ge 2\sqrt{8\cot\theta \cdot 2\tan\theta} = 2\sqrt{16}$

since $\tan\theta\cot\theta = 1$. Equality holds when $8\cot\theta = 2\tan\theta$, which is possible.

Minimum $= 2\sqrt{16} = 8$.

Ans: 8

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20. Using A.M. $\geq$ G.M., find the minimum value of $9\cos^2\theta + 25\sec^2\theta$, and say where it is reached. [5]

$9\cos^2\theta + 25\sec^2\theta = (3)^2\cos^2\theta + (5)^2\sec^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $9\cos^2\theta + 25\sec^2\theta \geq 2\sqrt{9 \cdot 25} = 30$

Equality needs $9\cos^2\theta = 25\sec^2\theta$, that is $\cos^2\theta = \frac{5}{3}$.

But $\frac{5}{3} > 1$, so equality is impossible; the bound 30 is not reached.

With $u = \cos^2\theta \in (0, 1]$, the value $9u + \frac{25}{u}$ decreases as u increases up to 1.

Minimum at $\cos^2\theta = 1$: $9 + 25 = 34$.

Ans: Minimum 34, at $\cos^2\theta = 1$