EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 05

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. The fundamental period of $f(x)$ is $\frac{2\pi}{5}$. What is the period of $-f(x)$? [1 mark]

2. Write down the periods of $|\sin x|$ and $\sin x$. [1 mark]

3. Find the period of $\cot(-3x)$. [1 mark]

4. Find the period of $2\cot\left(\frac{5x}{2} - \frac{\pi}{4}\right) - 4$. [2 marks]

5. Find the period of $\operatorname{cosec} 3x + \cos\frac{3x}{2}$. [2 marks]

6. Find the period of $f(x) = \cos\frac{5x}{2} + \sec 5x - \tan 3x$. [5 marks]

7. Find the period of $\operatorname{cosec}^7 x$. [1 mark]

8. Find the period of $\sin^2\frac{3x}{2}$. [2 marks]

9. Find the period of $5\sin^4 x + 3\cos^4 x$. [2 marks]

10. Find the period of $\sin^4 5x + \cos^4 5x$. [3 marks]

11. Find the period of $4x - [4x]$. [1 mark]

12. Find the period of $f(x) = 2x - [2x] + \tan \pi x$. [3 marks]

13. Write the range of $|\sin x|$. [1 mark]

14. Find the range of $5 + 3\tan^2 x$. [2 marks]

15. Find the maximum and minimum values of $\cos x + 3\sin x + 5$. [2 marks]

16. Find the maximum and minimum values of $f(x) = -3\sin\left(x + \frac{\pi}{3}\right) + 3\sin x$. [5 marks]

17. Find the range of $\dfrac{1}{3 + 2\sin x}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{-5\cos 5x + 12\sin 5x - 11}$. [5 marks]

19. Find the minimum value of $8\tan\theta + 2\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $9\sin^2\theta + 1\,\mathrm{cosec}^2\,\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	$\frac{2\pi}{5}$
2	π and 2π
3	$\frac{\pi}{3}$
4	$\frac{2\pi}{5}$
5	$\frac{4\pi}{3}$
6	4π
7	2π
8	$\frac{2\pi}{3}$
9	π
10	$\frac{\pi}{10}$
11	$\frac{1}{4}$
12	1
13	$[0, 1]$
14	$[5, \infty)$
15	Maximum $\sqrt{10} + 5$, minimum $5 - \sqrt{10}$
16	Maximum 3, minimum -3
17	$[\frac{1}{5}, 1]$
18	Period $\frac{2\pi}{5}$; range $(-\infty, -\frac{1}{24}] \cup [\frac{1}{2}, \infty)$
19	8
20	Minimum 6, at $\sin^2\theta = \frac{1}{3}$

Solutions

1. The fundamental period of $f(x)$ is $\frac{2\pi}{5}$. What is the period of $-f(x)$? [1]

If f has period p, then $-f$ and $\frac{1}{f}$ have period p; multiplying by a constant or adding a constant does not change the period.

So the period of $-f(x)$ is $\frac{2\pi}{5}$.

Ans: $\frac{2\pi}{5}$

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2. Write down the periods of $|\sin x|$ and $\sin x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are π and 2π.

Ans: π **and** 2π

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3. Find the period of $\cot(-3x)$. [1]

The period of $\cot x$ is π; for $f(ax+b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{\pi}{3}$

Ans: $\frac{\pi}{3}$

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4. Find the period of $2\cot\left(\frac{5x}{2} - \frac{\pi}{4}\right) - 4$. [2]

The period of $\cot x$ is π, and here $a = \frac{5}{2}$.

The constants 2 and -4 do not change the period.

$\text{Period} = \frac{\pi}{\frac{5}{2}} = \frac{2\pi}{5}$

Ans: $\frac{2\pi}{5}$

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5. Find the period of $\operatorname{cosec} 3x + \cos\frac{3x}{2}$. [2]

$\text{Period of } \operatorname{cosec} 3x = \frac{2\pi}{3}, \quad \text{period of } \cos\frac{3x}{2} = \frac{4\pi}{3}$

$\text{L.C.M.}\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} = \frac{4\pi}{3}$

Period $= \frac{4\pi}{3}$.

Ans: $\frac{4\pi}{3}$

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6. Find the period of $f(x) = \cos\frac{5x}{2} + \sec 5x - \tan 3x$. [5]

$\text{Period of } \cos\frac{5x}{2} = \frac{2\pi}{\frac{5}{2}} = \frac{4\pi}{5}$

$\text{Period of } \sec 5x = \frac{2\pi}{5}$

$\text{Period of } \tan 3x = \frac{\pi}{3}$

Write each period as a fraction times π: $\frac{4}{5}\pi$, $\frac{2}{5}\pi$, $\frac{1}{3}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{4,2,1\}}{\text{G.C.D.}\{5,5,3\}}\,\pi = \frac{4}{1}\,\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 4π.

Ans: 4π

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7. Find the period of $\operatorname{cosec}^7 x$. [1]

$\operatorname{cosec}^n x$ has period π when n is even and 2π when n is odd; here $n = 7$ is odd.

Period $= 2\pi$.

Ans: 2π

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8. Find the period of $\sin^2 \frac{3x}{2}$. [2]

$\sin^2 x$ has period π (2 is even).

$\text{Period of } \sin^2 \frac{3x}{2} = \frac{\pi}{\frac{3}{2}} = \frac{2\pi}{3}$

Ans: $\frac{2\pi}{3}$

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9. Find the period of $5\sin^4 x + 3\cos^4 x$. [2]

$n = 4$ is even and the coefficients are unequal.

From the table for $a\sin^n x + b\cos^n x$, the period is π.

Ans: π

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10. Find the period of $\sin^4 5x + \cos^4 5x$. [3]

For even n, $a\sin^n x + b\cos^n x$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are equal, so with argument x the period would be $\frac{\pi}{2}$.

The argument is $5x$, so divide by 5:

$\text{Period} = \frac{\frac{\pi}{2}}{5} = \frac{\pi}{10}$

Ans: $\frac{\pi}{10}$

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11. Find the period of $4x - [4x]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{4}$

Ans: $\frac{1}{4}$

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12. Find the period of $f(x) = 2x - [2x] + \tan \pi x$. [3]

$\text{Period of } 2x - [2x] = \frac{1}{2}$

$\text{Period of } \tan \pi x = \frac{\pi}{\pi} = 1$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{2}, 1\right\} = \frac{\text{L.C.M.}\{1,1\}}{\text{G.C.D.}\{2,1\}} = 1$

Ans: 1

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13. Write the range of $|\sin x|$. [1]

From the table of ranges, $|\sin x|$ takes the values $[0, 1]$.

Ans: $[0, 1]$

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14. Find the range of $5 + 3\tan^2 x$. [2]

The range of $\tan^2 x$ is $[0, \infty)$.

$0 \le \tan^2 x < \infty$

Multiply by 3 and add 5.

Range $= [5, \infty)$.

Ans: $[5, \infty)$

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15. Find the maximum and minimum values of $\cos x + 3\sin x + 5$. [2]

$a = 1,\ b = 3,\ c = 5$

$\sqrt{a^2 + b^2} = \sqrt{1 + 9} = \sqrt{10}$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = \sqrt{10} + 5, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = 5 - \sqrt{10}$

Ans: Maximum $\sqrt{10} + 5$**, minimum** $5 - \sqrt{10}$

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16. Find the maximum and minimum values of $f(x) = -3\sin\left(x + \frac{\pi}{3}\right) + 3\sin x$. [5]

Expand with $\sin(x \pm \beta) = \sin x \cos\beta \pm \cos x \sin\beta$:

$f(x) = -\frac{3}{2}\sin x - \frac{3\sqrt{3}}{2}\cos x + 3\sin x$

$= \left(-\frac{3\sqrt{3}}{2}\right)\cos x + \left(\frac{3}{2}\right)\sin x$

$a^2 + b^2 = \frac{27}{4} + \frac{9}{4} = 9$, so $\sqrt{a^2 + b^2} = 3$

$\text{Maximum} = 0 + 3 = 3, \quad \text{minimum} = 0 - 3 = -3$

Ans: Maximum 3, minimum -3

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17. Find the range of $\dfrac{1}{3 + 2\sin x}$. [3]

$-1 \le \sin x \le 1 \Rightarrow 1 \le 3 + 2\sin x \le 5$

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range $= [\frac{1}{5}, 1]$.

Ans: $[\frac{1}{5}, 1]$

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18. Find the period and the range of $f(x) = \dfrac{1}{-5\cos 5x + 12\sin 5x - 11}$. [5]

The denominator $g(x) = -5\cos 5x + 12\sin 5x - 11$ has period $\frac{2\pi}{5} = \frac{2\pi}{5}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{25 + 144} = 13$

Range of g: $[-11 - 13, -11 + 13] = [-24, 2]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range of f: $(-\infty, -\frac{1}{24}] \cup [\frac{1}{2}, \infty)$.

Ans: Period $\frac{2\pi}{5}$**; range** $(-\infty, -\frac{1}{24}] \cup [\frac{1}{2}, \infty)$

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19. Find the minimum value of $8\tan\theta + 2\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$8\tan\theta + 2\cot\theta \ge 2\sqrt{8\tan\theta \cdot 2\cot\theta} = 2\sqrt{16}$

since $\tan\theta\cot\theta = 1$. Equality holds when $8\tan\theta = 2\cot\theta$, which is possible.

Minimum $= 2\sqrt{16} = 8$.

Ans: 8

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20. Using A.M. $\geq$ G.M., find the minimum value of $9\sin^2\theta + 1\,\mathrm{cosec}^2\theta$, and say where it is reached. [5]

$9\sin^2\theta + 1\,\mathrm{cosec}^2\theta = (3)^2\sin^2\theta + (1)^2\,\mathrm{cosec}^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $9\sin^2\theta + 1\,\mathrm{cosec}^2\theta \geq 2\sqrt{9 \cdot 1} = 6$

Equality needs $9\sin^2\theta = 1\,\mathrm{cosec}^2\theta$, that is $\sin^2\theta = \frac{1}{3}$.

Since $\frac{1}{3} \leq 1$, this is possible.

Minimum $= 6$, reached when $\sin^2\theta = \frac{1}{3}$.

Ans: Minimum 6, at $\sin^2\theta = \frac{1}{3}$
