EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. f is a periodic function with period 4, and $f(-1) = 2$. Find $f(11)$. [1 mark]

2. Write down the periods of $\cot x$ and $|\tan x|$. [1 mark]

3. Find the period of $\sin(10x + 7)$. [1 mark]

4. Find the period of $7\cot\left(\frac{x}{4} + 1\right) - 1$. [2 marks]

5. Find the period of $\cot\frac{x}{2} + \cot\frac{2x}{3}$. [2 marks]

6. Find the period of $f(x) = \operatorname{cosec} 3x - \tan\frac{x}{4} + \sin\frac{3x}{2}$. [5 marks]

7. Find the period of $\tan^2 x$. [1 mark]

8. Find the period of $\tan^4\frac{3x}{4}$. [2 marks]

9. Find the period of $6\tan^4 x + 6\cot^4 x$. [2 marks]

10. Find the period of $5\sin^4\frac{x}{2} + 2\cos^4\frac{x}{2}$. [3 marks]

11. Find the period of $\frac{3x}{4} - \left[\frac{3x}{4}\right]$. [1 mark]

12. Find the period of $f(x) = 2x - [2x] + \sin\frac{2\pi x}{3}$. [3 marks]

13. Write the range of $\cos^2 x$. [1 mark]

14. Find the range of $7 + 2\sin^5 x$. [2 marks]

15. Find the maximum and minimum values of $7\sin x + 24\cos x - 4$. [2 marks]

16. Find the maximum and minimum values of $f(x) = 6\cos\left(x - \frac{\pi}{3}\right) - 6\cos x + 1$. [5 marks]

17. Find the range of $\dfrac{1}{4\cos x + 3\sin x - 1}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{-4\cos 6x + 3\sin 6x + 4}$. [5 marks]

19. Find the minimum value of $16\cot\theta + 9\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $16\cos^2\theta + 49\sec^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	2
2	π and π
3	$\frac{\pi}{5}$
4	4π
5	6π
6	4π
7	π
8	$\frac{4\pi}{3}$
9	$\frac{\pi}{2}$
10	2π
11	$\frac{4}{3}$
12	3
13	$[0, 1]$
14	$[5, 9]$
15	Maximum 21, minimum -29
16	Maximum 7, minimum -5
17	$(-\infty, -\frac{1}{6}] \cup [\frac{1}{4}, \infty)$
18	Period $\frac{\pi}{3}$; range $(-\infty, -1] \cup [\frac{1}{9}, \infty)$
19	24
20	Minimum 65, at $\cos^2\theta = 1$

Solutions

1. f is a periodic function with period 4, and $f(-1) = 2$. Find $f(11)$. [1]

$11 = -1 + 3 \times 4$

$f(x + np) = f(x) \Rightarrow f(11) = f(-1) = 2$

Ans: 2

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2. Write down the periods of $\cot x$ and $|\tan x|$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are π and π.

Ans: π **and** π

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3. Find the period of $\sin(10x + 7)$. [1]

The period of $\sin x$ is 2π; for $f(ax + b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{2\pi}{10} = \frac{\pi}{5}$

Ans: $\frac{\pi}{5}$

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4. Find the period of $7\cot\left(\frac{x}{4} + 1\right) - 1$. [2]

The period of $\cot x$ is π, and here $a = \frac{1}{4}$.

The constants 7 and -1 do not change the period.

$\text{Period} = \frac{\pi}{\frac{1}{4}} = 4\pi$

Ans: 4π

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5. Find the period of $\cot\frac{x}{2} + \cot\frac{2x}{3}$. [2]

$\text{Period of } \cot\frac{x}{2} = 2\pi, \quad \text{period of } \cot\frac{2x}{3} = \frac{3\pi}{2}$

$\text{L.C.M.}\left\{2\pi, \frac{3\pi}{2}\right\} = 6\pi$

Period $= 6\pi$.

Ans: 6π

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6. Find the period of $f(x) = \operatorname{cosec} 3x - \tan\frac{x}{4} + \sin\frac{3x}{2}$. [5]

$\text{Period of } \operatorname{cosec} 3x = \frac{2\pi}{3}$

$\text{Period of } \tan\frac{x}{4} = \frac{\pi}{\frac{1}{4}} = 4\pi$

$\text{Period of } \sin\frac{3x}{2} = \frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$

Write each period as a fraction times π: $\frac{2}{3}\pi$, $\frac{4}{1}\pi$, $\frac{4}{3}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{2,4,4\}}{\text{G.C.D.}\{3,1,3\}}\,\pi = \frac{4}{1}\,\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 4π.

Ans: 4π

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7. Find the period of $\tan^2 x$. [1]

$\tan^n x$ and $\cot^n x$ have period π for every n.

Period $= \pi$.

Ans: π

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8. Find the period of $\tan^4 \frac{3x}{4}$. [2]

$\tan^4 x$ has period π (tangent family).

$\text{Period of } \tan^4 \frac{3x}{4} = \frac{\pi}{\frac{3}{4}} = \frac{4\pi}{3}$

Ans: $\frac{4\pi}{3}$

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9. Find the period of $6\tan^4 x + 6\cot^4 x$. [2]

$n = 4$ is even and the coefficients are equal.

Shifting x by $\frac{\pi}{2}$ swaps the two terms, so the sum repeats after $\frac{\pi}{2}$.

From the table for $a\tan^n x + b\cot^n x$, the period is $\frac{\pi}{2}$.

Ans: $\frac{\pi}{2}$

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10. Find the period of $5\sin^4 \frac{x}{2} + 2\cos^4 \frac{x}{2}$. [3]

For even n, $a\sin^n x + b\cos^n x$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are unequal, so with argument x the period would be π.

The argument is $\frac{x}{2}$, so divide by $\frac{1}{2}$:

$\text{Period} = \frac{\pi}{\frac{1}{2}} = 2\pi$

Ans: 2π

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11. Find the period of $\frac{3x}{4} - \left[\frac{3x}{4}\right]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

Ans: $\frac{4}{3}$

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12. Find the period of $f(x) = 2x - [2x] + \sin \frac{2\pi x}{3}$. [3]

$\text{Period of } 2x - [2x] = \frac{1}{2}$

$\text{Period of } \sin \frac{2\pi x}{3} = \frac{2\pi}{\frac{2\pi}{3}} = 3$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{2}, 3\right\} = \frac{\text{L.C.M.}\{1,3\}}{\text{G.C.D.}\{2,1\}} = 3$

Ans: 3

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13. Write the range of $\cos^2 x$. [1]

From the table of ranges, $\cos^2 x$ takes the values $[0, 1]$.

Ans: $[0, 1]$

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14. Find the range of $7 + 2\sin^5 x$. [2]

The range of $\sin^5 x$ is $[-1, 1]$.

$-1 \leq \sin^5 x \leq 1$

Multiply by 2 and add 7.

Range $= [5, 9]$.

Ans: $[5, 9]$

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15. Find the maximum and minimum values of $7\sin x + 24\cos x - 4$. [2]

$a = 24,\ b = 7,\ c = -4$

$\sqrt{a^2 + b^2} = \sqrt{576 + 49} = 25$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 21, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -29$

Ans: Maximum 21, minimum -29

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16. Find the maximum and minimum values of $f(x) = 6\cos\left(x - \frac{\pi}{3}\right) - 6\cos x + 1$. [5]

Expand with $\cos(x \pm \beta) = \cos x \cos\beta \mp \sin x \sin\beta$:

$f(x) = 3\sqrt{3}\sin x + 3\cos x - 6\cos x + 1$

$= (-3)\cos x + \left(3\sqrt{3}\right)\sin x + 1$

$a^2 + b^2 = 9 + 27 = 36$, so $\sqrt{a^2 + b^2} = 6$

$\text{Maximum} = 1 + 6 = 7, \quad \text{minimum} = 1 - 6 = -5$

Ans: Maximum 7, minimum -5

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17. Find the range of $\dfrac{1}{4\cos x + 3\sin x - 1}$. [3]

$\sqrt{16 + 9} = 5$

The denominator has range $[-6, 4]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range $= (-\infty, -\frac{1}{6}] \cup [\frac{1}{4}, \infty)$.

Ans: $(-\infty, -\frac{1}{6}] \cup [\frac{1}{4}, \infty)$

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18. Find the period and the range of $f(x) = \dfrac{1}{-4\cos 6x + 3\sin 6x + 4}$. [5]

The denominator $g(x) = -4\cos 6x + 3\sin 6x + 4$ has period $\frac{2\pi}{6} = \frac{\pi}{3}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

Range of g: $[4 - 5, 4 + 5] = [-1, 9]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range of f: $(-\infty, -1] \cup [\frac{1}{9}, \infty)$.

Ans: Period $\frac{\pi}{3}$; range $(-\infty, -1] \cup [\frac{1}{9}, \infty)$

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19. Find the minimum value of $16\cot\theta + 9\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\geq$ G.M.:

$16\cot\theta + 9\tan\theta \geq 2\sqrt{16\cot\theta \cdot 9\tan\theta} = 2\sqrt{144}$

since $\tan\theta\cot\theta = 1$. Equality holds when $16\cot\theta = 9\tan\theta$, which is possible.

Minimum $= 2\sqrt{144} = 24$.

Ans: 24

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20. Using A.M. $\geq$ G.M., find the minimum value of $16\cos^2\theta + 49\sec^2\theta$, and say where it is reached. [5]

$16\cos^2\theta + 49\sec^2\theta = (4)^2\cos^2\theta + (7)^2\sec^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $16\cos^2\theta + 49\sec^2\theta \geq 2\sqrt{16 \cdot 49} = 56$

Equality needs $16\cos^2\theta = 49\sec^2\theta$, that is $\cos^2\theta = \frac{7}{4}$.

But $\frac{7}{4} > 1$, so equality is impossible; the bound 56 is not reached.

With $u = \cos^2\theta \in (0, 1]$, the value $16u + \frac{49}{u}$ decreases as u increases up to 1.

Minimum at $\cos^2\theta = 1$: $16 + 49 = 65$.

Ans: Minimum 65, at $\cos^2\theta = 1$