EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Show that the constant function $f(x) = \frac{1}{2}$ is periodic, and say whether it has a fundamental period. [1 mark]

2. Write down the periods of $|\sin x|$ and $\tan x$. [1 mark]

3. Find the period of $\sin(8x + 3)$. [1 mark]

4. Find the period of $5\cot\left(2\pi x + \frac{\pi}{6}\right) + 5$. [2 marks]

5. Find the period of $\operatorname{cosec}\frac{3x}{2} - \cot 4x$. [2 marks]

6. Find the period of $f(x) = \cos\frac{3x}{4} - \sec\frac{5x}{4} + \sec\frac{3x}{2}$. [5 marks]

7. Find the period of $\operatorname{cosec}^5 x$. [1 mark]

8. Find the period of $\sec^6 6x$. [2 marks]

9. Find the period of $6\tan^3 x + 6\cot^3 x$. [2 marks]

10. Find the period of $|\sin\frac{x}{3}| + |\cos\frac{x}{3}|$. [3 marks]

11. Find the period of $5x - [5x]$. [1 mark]

12. Find the period of $f(x) = 6x - [6x] + \sec\frac{\pi x}{2}$. [3 marks]

13. Write the range of $\cos^6 x$. [1 mark]

14. Find the range of $3 - \operatorname{cosec} x$. [2 marks]

15. Find the maximum and minimum values of $4\cos x + 3\sin x + 3$. [2 marks]

16. Find the maximum and minimum values of $f(x) = 9\cos\left(x + \frac{\pi}{3}\right) - 9\cos x + 4$. [5 marks]

17. Find the range of $\dfrac{1}{-4 + 2\cos x}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{4\cos 2x + 3\sin 2x + 1}$. [5 marks]

19. Find the minimum value of $4\tan\theta + 25\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $4\sin^2\theta + 36\,\mathrm{cosec}^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	Periodic; no fundamental period
2	π and π
3	$\frac{\pi}{4}$
4	$\frac{1}{2}$
5	4π
6	8π
7	2π
8	$\frac{\pi}{6}$
9	π
10	$\frac{3\pi}{2}$
11	$\frac{1}{5}$
12	4
13	$[0, 1]$
14	$(-\infty, 2] \cup [4, \infty)$
15	Maximum 8, minimum -2
16	Maximum 13, minimum -5
17	$[-\frac{1}{2}, -\frac{1}{6}]$
18	Period π; range $(-\infty, -\frac{1}{4}] \cup [\frac{1}{6}, \infty)$
19	20
20	Minimum 40, at $\sin^2\theta = 1$

Solutions

1. Show that the constant function $f(x) = \frac{1}{2}$ is periodic, and say whether it has a fundamental period. [1]

$f(x+p) = \frac{1}{2} = f(x)$ for every positive number p.

So f is periodic.

There is no least positive number, so f has no fundamental period.

Ans: Periodic; no fundamental period

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2. Write down the periods of $|\sin x|$ and $\tan x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are π and π.

Ans: π and π

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3. Find the period of $\sin(8x+3)$. [1]

The period of $\sin x$ is 2π; for $f(ax+b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{2\pi}{8} = \frac{\pi}{4}$

Ans: $\frac{\pi}{4}$

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4. Find the period of $5\cot\left(2\pi x + \frac{\pi}{6}\right) + 5$. [2]

The period of $\cot x$ is π, and here $a = 2\pi$.

The constants 5 and 5 do not change the period.

$\text{Period} = \frac{\pi}{2\pi} = \frac{1}{2}$

Ans: $\frac{1}{2}$

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5. Find the period of $\operatorname{cosec}\frac{3x}{2} - \cot 4x$. [2]

$\text{Period of } \operatorname{cosec}\frac{3x}{2} = \frac{4\pi}{3}, \quad \text{period of } \cot 4x = \frac{\pi}{4}$

$\text{L.C.M.}\left\{\frac{4\pi}{3}, \frac{\pi}{4}\right\} = 4\pi$

Period $= 4\pi$.

Ans: 4π

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6. Find the period of $f(x) = \cos\frac{3x}{4} - \sec\frac{5x}{4} + \sec\frac{3x}{2}$. [5]

$\text{Period of } \cos\frac{3x}{4} = \frac{2\pi}{\frac{3}{4}} = \frac{8\pi}{3}$

$\text{Period of } \sec\frac{5x}{4} = \frac{2\pi}{\frac{5}{4}} = \frac{8\pi}{5}$

$\text{Period of } \sec\frac{3x}{2} = \frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$

Write each period as a fraction times π: $\frac{8}{3}\pi$, $\frac{8}{5}\pi$, $\frac{4}{3}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{8,8,4\}}{\text{G.C.D.}\{3,5,3\}}\,\pi = \frac{8}{1}\,\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 8π.

Ans: 8π

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7. Find the period of $\operatorname{cosec}^5 x$. [1]

$\operatorname{cosec}^n x$ has period π when n is even and 2π when n is odd; here $n = 5$ is odd.

Period $= 2\pi$.

Ans: 2π

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8. Find the period of $\sec^6 6x$. [2]

$\sec^6 x$ has period π (6 is even).

$\text{Period of } \sec^6 6x = \frac{\pi}{6}$

Ans: $\frac{\pi}{6}$

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9. Find the period of $6\tan^3 x + 6\cot^3 x$. [2]

$n = 3$ is odd.

From the table for $a\tan^n x + b\cot^n x$, the period is π.

Ans: π

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10. Find the period of $|\sin\frac{x}{3}| + |\cos\frac{x}{3}|$. [3]

$a|\sin x| + b|\cos x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are equal, so with argument x the period would be $\frac{\pi}{2}$.

The argument is $\frac{x}{3}$, so divide by $\frac{1}{3}$:

$\text{Period} = \frac{\frac{\pi}{2}}{\frac{1}{3}} = \frac{3\pi}{2}$

Ans: $\frac{3\pi}{2}$

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11. Find the period of $5x - [5x]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{5}$

Ans: $\frac{1}{5}$

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12. Find the period of $f(x) = 6x - [6x] + \sec\frac{\pi x}{2}$. [3]

$\text{Period of } 6x - [6x] = \frac{1}{6}$

$\text{Period of } \sec\frac{\pi x}{2} = \frac{2\pi}{\frac{\pi}{2}} = 4$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{6}, 4\right\} = \frac{\text{L.C.M.}\{1,4\}}{\text{G.C.D.}\{6,1\}} = 4$

Ans: 4

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13. Write the range of $\cos^6 x$. [1]

From the table of ranges, $\cos^6 x$ takes the values $[0, 1]$.

Ans: $[0, 1]$

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14. Find the range of $3 - \operatorname{cosec} x$. [2]

The range of $\operatorname{cosec} x$ is $(-\infty, -1] \cup [1, \infty)$.

Multiply by -1 and add 3 on each piece (a negative factor reverses the order).

Range $= (-\infty, 2] \cup [4, \infty)$.

Ans: $(-\infty, 2] \cup [4, \infty)$

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15. Find the maximum and minimum values of $4\cos x + 3\sin x + 3$. [2]

$a = 4,\ b = 3,\ c = 3$

$\sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 8, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -2$

Ans: Maximum 8, minimum -2

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16. Find the maximum and minimum values of $f(x) = 9\cos\left(x + \frac{\pi}{3}\right) - 9\cos x + 4$. [5]

Expand with $\cos(x \pm \beta) = \cos x \cos\beta \mp \sin x \sin\beta$:

$f(x) = -\frac{9\sqrt{3}}{2}\sin x + \frac{9}{2}\cos x - 9\cos x + 4$

$= \left(-\frac{9}{2}\right)\cos x + \left(-\frac{9\sqrt{3}}{2}\right)\sin x + 4$

$a^2 + b^2 = \frac{81}{4} + \frac{243}{4} = 81$, so $\sqrt{a^2 + b^2} = 9$

$\text{Maximum} = 4 + 9 = 13, \quad \text{minimum} = 4 - 9 = -5$

Ans: Maximum 13, minimum -5

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17. Find the range of $\dfrac{1}{-4 + 2\cos x}$. [3]

$-1 \le \cos x \le 1 \Rightarrow -6 \le -4 + 2\cos x \le -2$

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range $= [-\frac{1}{2}, -\frac{1}{6}]$.

Ans: $[-\frac{1}{2}, -\frac{1}{6}]$

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18. Find the period and the range of $f(x) = \dfrac{1}{4\cos 2x + 3\sin 2x + 1}$. [5]

The denominator $g(x) = 4\cos 2x + 3\sin 2x + 1$ has period $\frac{2\pi}{2} = \pi$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

Range of g: $[1 - 5, 1 + 5] = [-4, 6]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range of f: $(-\infty, -\frac{1}{4}] \cup [\frac{1}{6}, \infty)$.

Ans: Period π; range $(-\infty, -\frac{1}{4}] \cup [\frac{1}{6}, \infty)$

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19. Find the minimum value of $4\tan\theta + 25\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$4\tan\theta + 25\cot\theta \ge 2\sqrt{4\tan\theta \cdot 25\cot\theta} = 2\sqrt{100}$

since $\tan\theta\cot\theta = 1$. Equality holds when $4\tan\theta = 25\cot\theta$, which is possible.

Minimum $= 2\sqrt{100} = 20$.

Ans: 20

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20. Using A.M. $\geq$ G.M., find the minimum value of $4\sin^2\theta + 36\,\mathrm{cosec}^2\theta$, and say where it is reached. [5]

$4\sin^2\theta + 36\,\mathrm{cosec}^2\theta = (2)^2\sin^2\theta + (6)^2\,\mathrm{cosec}^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $4\sin^2\theta + 36\,\mathrm{cosec}^2\theta \geq 2\sqrt{4 \cdot 36} = 24$

Equality needs $4\sin^2\theta = 36\,\mathrm{cosec}^2\theta$, that is $\sin^2\theta = 3$.

But $3 > 1$, so equality is impossible; the bound 24 is not reached.

With $u = \sin^2\theta \in (0, 1]$, the value $4u + \frac{36}{u}$ decreases as u increases up to 1.

Minimum at $\sin^2\theta = 1$: $4 + 36 = 40$.

Ans: Minimum 40**, at** $\sin^2\theta = 1$