EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. The fundamental period of $f(x)$ is $\frac{3}{2}$. What is the period of $f(x) + 3$? [1 mark]

2. Write down the periods of $\sin x$ and $\cos x$. [1 mark]

3. Find the period of $\tan(5x + 7)$. [1 mark]

4. Find the period of $-3\cos\left(\frac{x}{2} + 2\right) + 2$. [2 marks]

5. Find the period of $\operatorname{cosec}\frac{2x}{3} - \cot 4x$. [2 marks]

6. Find the period of $f(x) = \cot\frac{x}{2} - \tan\frac{2x}{3} - \sin\frac{x}{4}$. [5 marks]

7. Find the period of $\cos^4 x$. [1 mark]

8. Find the period of $\cos^6\frac{x}{2}$. [2 marks]

9. Find the period of $5\operatorname{cosec}^6 x + 5\sec^6 x$. [2 marks]

10. Find the period of $2|\tan\frac{x}{2}| + |\cot\frac{x}{2}|$. [3 marks]

11. Find the period of $7x - [7x]$. [1 mark]

12. Find the period of $f(x) = \frac{x}{3} - \left[\frac{x}{3}\right] + \sec\frac{\pi x}{3}$. [3 marks]

13. Write the range of $\operatorname{cosec}^2 x$. [1 mark]

14. Find the range of $4 - \sec x$. [2 marks]

15. Find the maximum and minimum values of $7\sin x + 24\cos x + 7$. [2 marks]

16. Find the maximum and minimum values of $f(x) = -7\sin\left(x - \frac{\pi}{6}\right) + 8\cos x$. [5 marks]

17. Find the range of $\dfrac{1}{4\cos x - 3\sin x - 5}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{-12\cos 5x + 5\sin 5x + 16}$. [5 marks]

19. Find the minimum value of $9\cot\theta + 25\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $16\cos^2\theta + 25\sec^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	$\frac{3}{2}$
2	2π and 2π
3	$\frac{\pi}{5}$
4	4π
5	3π
6	24π
7	π
8	2π
9	$\frac{\pi}{2}$
10	2π
11	$\frac{1}{7}$
12	6
13	$[1, \infty)$
14	$(-\infty, 3] \cup [5, \infty)$
15	Maximum 32, minimum -18
16	Maximum 13, minimum -13
17	$(-\infty, -\frac{1}{10}]$
18	Period $\frac{2\pi}{5}$; range $[\frac{1}{29}, \frac{1}{3}]$
19	30
20	Minimum 41, at $\cos^2\theta = 1$

Solutions

1. The fundamental period of $f(x)$ is $\frac{3}{2}$. What is the period of $f(x)+3$? [1]

If f has period p, then $-f$ and $\frac{1}{f}$ have period p; multiplying by a constant or adding a constant does not change the period.

So the period of $f(x)+3$ is $\frac{3}{2}$.

Ans: $\frac{3}{2}$

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2. Write down the periods of $\sin x$ and $\cos x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are 2π and 2π.

Ans: 2π **and** 2π

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3. Find the period of $\tan(5x+7)$. [1]

The period of $\tan x$ is π; for $f(ax+b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{\pi}{5}$

Ans: $\frac{\pi}{5}$

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4. Find the period of $-3\cos\left(\frac{x}{2}+2\right)+2$. [2]

The period of $\cos x$ is 2π, and here $a=\frac{1}{2}$.

The constants -3 and 2 do not change the period.

$\text{Period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Ans: 4π

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5. Find the period of $\operatorname{cosec}\frac{2x}{3} - \cot 4x$. [2]

$\text{Period of } \operatorname{cosec}\frac{2x}{3} = 3\pi, \quad \text{period of } \cot 4x = \frac{\pi}{4}$

$\text{L.C.M.}\left\{3\pi, \frac{\pi}{4}\right\} = 3\pi$

Period $= 3\pi$.

Ans: 3π

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6. Find the period of $f(x) = \cot\frac{x}{2} - \tan\frac{2x}{3} - \sin\frac{x}{4}$. [5]

$\text{Period of } \cot\frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$

$\text{Period of } \tan\frac{2x}{3} = \frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}$

$\text{Period of } \sin\frac{x}{4} = \frac{2\pi}{\frac{1}{4}} = 8\pi$

Write each period as a fraction times π: $\frac{2}{1}\pi$, $\frac{3}{2}\pi$, $\frac{8}{1}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{2,3,8\}}{\text{G.C.D.}\{1,2,1\}}\pi = \frac{24}{1}\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 24π.

Ans: 24π

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7. Find the period of $\cos^4 x$. [1]

$\cos^n x$ has period π when n is even and 2π when n is odd; here $n = 4$ is even.

Period $= \pi$.

Ans: π

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8. Find the period of $\cos^6 \frac{x}{2}$. [2]

$\cos^6 x$ has period π (6 is even).

$\text{Period of } \cos^6 \frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$

Ans: 2π

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9. Find the period of $5\,\text{cosec}^6 x + 5\sec^6 x$. [2]

$n = 6$ is even and the coefficients are equal.

Shifting x by $\frac{\pi}{2}$ swaps the two terms, so the sum repeats after $\frac{\pi}{2}$.

From the table for $a\,\text{cosec}^n x + b\sec^n x$, the period is $\frac{\pi}{2}$.

Ans: $\frac{\pi}{2}$

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10. Find the period of $2|\tan \frac{x}{2}| + |\cot \frac{x}{2}|$. [3]

$a|\tan x| + b|\cot x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are unequal, so with argument x the period would be π.

The argument is $\frac{x}{2}$, so divide by $\frac{1}{2}$:

$\text{Period} = \frac{\pi}{\frac{1}{2}} = 2\pi$

Ans: 2π

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11. Find the period of $7x - [7x]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{7}$

Ans: $\frac{1}{7}$

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12. Find the period of $f(x) = \frac{x}{3} - [\frac{x}{3}] + \sec \frac{\pi x}{3}$. [3]

$\text{Period of } \frac{x}{3} - [\frac{x}{3}] = \frac{1}{\frac{1}{3}} = 3$

$\text{Period of } \sec \frac{\pi x}{3} = \frac{2\pi}{\frac{\pi}{3}} = 6$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\,\{3, 6\} = \frac{\text{L.C.M.}\{3,6\}}{\text{G.C.D.}\{1,1\}} = 6$

Ans: 6

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13. Write the range of $\text{cosec}^2 x$. [1]

From the table of ranges, $\text{cosec}^2 x$ takes the values $[1, \infty)$.

Ans: $[1, \infty)$

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14. Find the range of $4 - \sec x$. [2]

The range of $\sec x$ is $(-\infty, -1] \cup [1, \infty)$.

Multiply by -1 and add 4 on each piece (a negative factor reverses the order).

Range $= (-\infty, 3] \cup [5, \infty)$.

Ans: $(-\infty, 3] \cup [5, \infty)$

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15. Find the maximum and minimum values of $7\sin x + 24\cos x + 7$. [2]

$a = 24,\ b = 7,\ c = 7$

$\sqrt{a^2 + b^2} = \sqrt{576 + 49} = 25$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 32, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -18$

Ans: Maximum 32**, minimum** -18

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16. Find the maximum and minimum values of $f(x) = -7\sin\left(x - \frac{\pi}{6}\right) + 8\cos x$. [5]

Expand with $\sin(x \pm \beta) = \sin x \cos\beta \pm \cos x \sin\beta$:

$f(x) = -\frac{7\sqrt{3}}{2}\sin x + \frac{7}{2}\cos x + 8\cos x$

$= \left(\frac{23}{2}\right)\cos x + \left(-\frac{7\sqrt{3}}{2}\right)\sin x$

$a^2 + b^2 = \frac{529}{4} + \frac{147}{4} = 169$, so $\sqrt{a^2 + b^2} = 13$

$\text{Maximum} = 0 + 13 = 13, \quad \text{minimum} = 0 - 13 = -13$

Ans: Maximum 13**, minimum** -13

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17. Find the range of $\dfrac{1}{4\cos x - 3\sin x - 5}$. [3]

$\sqrt{16 + 9} = 5$

The denominator has range $[-10, 0]$.

The range of f is $[a, 0]$, so the range of $\frac{1}{f}$ (where defined) is $\left(-\infty, \frac{1}{a}\right]$.

Range $= (-\infty, -\frac{1}{10}]$.

Ans: $(-\infty, -\frac{1}{10}]$

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18. Find the period and the range of $f(x) = \dfrac{1}{-12\cos 5x + 5\sin 5x + 16}$. [5]

The denominator $g(x) = -12\cos 5x + 5\sin 5x + 16$ has period $\frac{2\pi}{5} = \frac{2\pi}{5}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{144 + 25} = 13$

Range of g: $[16 - 13, 16 + 13] = [3, 29]$.

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range of f: $[\frac{1}{29}, \frac{1}{3}]$.

Ans: Period $\frac{2\pi}{5}$**; range** $[\frac{1}{29}, \frac{1}{3}]$

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19. Find the minimum value of $9\cot\theta + 25\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\geq$ G.M.:

$9\cot\theta + 25\tan\theta \geq 2\sqrt{9\cot\theta \cdot 25\tan\theta} = 2\sqrt{225}$

since $\tan\theta\cot\theta = 1$. Equality holds when $9\cot\theta = 25\tan\theta$, which is possible.

Minimum $= 2\sqrt{225} = 30$.

Ans: 30

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20. Using A.M. $\geq$ G.M., find the minimum value of $16\cos^2\theta + 25\sec^2\theta$, and say where it is reached. [5]

$16\cos^2\theta + 25\sec^2\theta = (4)^2\cos^2\theta + (5)^2\sec^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $16\cos^2\theta + 25\sec^2\theta \geq 2\sqrt{16 \cdot 25} = 40$

Equality needs $16\cos^2\theta = 25\sec^2\theta$, that is $\cos^2\theta = \frac{5}{4}$.

But $\frac{5}{4} > 1$, so equality is impossible; the bound 40 is not reached.

With $u = \cos^2\theta \in (0, 1]$, the value $16u + \frac{25}{u}$ decreases as u increases up to 1.

Minimum at $\cos^2\theta = 1$: $16 + 25 = 41$.

Ans: Minimum 41, at $\cos^2\theta = 1$
