EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. f is a periodic function with period 3, and $f(4) = 8$. Find $f(-5)$. [1 mark]

2. Write down the periods of $|\cos x|$ and $\operatorname{cosec} x$. [1 mark]

3. Find the period of $\cos(2x + 5)$. [1 mark]

4. Find the period of $4\tan\left(\frac{2x}{5} + 7\right) + 1$. [2 marks]

5. Find the period of $\operatorname{cosec} 4x - \tan 6x$. [2 marks]

6. Find the period of $f(x) = \cos\frac{3x}{4} - \tan\frac{x}{4} - \cot\frac{x}{2}$. [5 marks]

7. Find the period of $\sin^3 x$. [1 mark]

8. Find the period of $\sin^6\frac{2x}{3}$. [2 marks]

9. Find the period of $2\sin^3 x + 2\cos^3 x$. [2 marks]

10. Find the period of $2|\sin 2x| + 6|\cos 2x|$. [3 marks]

11. Find the period of $\frac{5x}{2} - \left[\frac{5x}{2}\right]$. [1 mark]

12. Find the period of $f(x) = 2x - [2x] + \sec 5\pi x$. [3 marks]

13. What is the range of $\cos^3 x$? [1 mark]

14. Find the range of $4 + 2\tan^2 x$. [2 marks]

15. Find the maximum and minimum values of $8\sin x + 6\cos x - 2$. [2 marks]

16. Find the maximum and minimum values of $f(x) = -3\sin\left(x - \frac{\pi}{6}\right) - 8\cos x + 5$. [5 marks]

17. Find the range of $\dfrac{1}{6\cos x + 8\sin x + 2}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{7\cos 6x - 24\sin 6x + 28}$. [5 marks]

19. Find the minimum value of $4\cot\theta + 9\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $25\cos^2\theta + 49\sec^2\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	8
2	π and 2π
3	π
4	$\frac{5\pi}{2}$
5	$\frac{\pi}{2}$
6	8π
7	2π
8	$\frac{3\pi}{2}$
9	2π
10	$\frac{\pi}{2}$
11	$\frac{2}{5}$
12	2
13	$[-1, 1]$
14	$[4, \infty)$
15	Maximum 8, minimum -12
16	Maximum 12, minimum -2
17	$(-\infty, -\frac{1}{8}] \cup [\frac{1}{12}, \infty)$
18	Period $\frac{\pi}{3}$; range $[\frac{1}{53}, \frac{1}{3}]$
19	12
20	Minimum 74, at $\cos^2\theta = 1$

Solutions

1. f is a periodic function with period 3, and $f(4) = 8$. Find $f(-5)$. [1]

$-5 = 4 - 3 \times 3$

$f(x + np) = f(x) \Rightarrow f(-5) = f(4) = 8$

Ans: 8

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2. Write down the periods of $|\cos x|$ and $\operatorname{cosec} x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are π and 2π.

Ans: π **and** 2π

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3. Find the period of $\cos(2x + 5)$. [1]

The period of $\cos x$ is 2π; for $f(ax + b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{2\pi}{2} = \pi$

Ans: π

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4. Find the period of $4\tan\left(\frac{2x}{5} + 7\right) + 1$. [2]

The period of $\tan x$ is π, and here $a = \frac{2}{5}$.

The constants 4 and 1 do not change the period.

$\text{Period} = \frac{\pi}{\frac{2}{5}} = \frac{5\pi}{2}$

Ans: $\frac{5\pi}{2}$

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5. Find the period of $\operatorname{cosec} 4x - \tan 6x$. [2]

$\text{Period of } \operatorname{cosec} 4x = \frac{\pi}{2}, \quad \text{period of } \tan 6x = \frac{\pi}{6}$

$\text{L.C.M.} \left\{\frac{\pi}{2}, \frac{\pi}{6}\right\} = \frac{\pi}{2}$

Period $= \frac{\pi}{2}$.

Ans: $\frac{\pi}{2}$

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6. Find the period of $f(x) = \cos\frac{3x}{4} - \tan\frac{x}{4} - \cot\frac{x}{2}$. [5]

$\text{Period of } \cos\frac{3x}{4} = \frac{2\pi}{\frac{3}{4}} = \frac{8\pi}{3}$

$\text{Period of } \tan\frac{x}{4} = \frac{\pi}{\frac{1}{4}} = 4\pi$

$\text{Period of } \cot\frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$

Write each period as a fraction times π: $\frac{8}{3}\pi$, $\frac{4}{1}\pi$, $\frac{2}{1}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{8,4,2\}}{\text{G.C.D.}\{3,1,1\}}\pi = \frac{8}{1}\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 8π.

Ans: 8π

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7. Find the period of $\sin^3 x$. [1]

$\sin^n x$ has period π when n is even and 2π when n is odd; here $n = 3$ is odd.

Period $= 2\pi$.

Ans: 2π

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8. Find the period of $\sin^6 \frac{2x}{3}$. [2]

$\sin^6 x$ has period π (6 is even).

$\text{Period of } \sin^6 \frac{2x}{3} = \frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}$

Ans: $\frac{3\pi}{2}$

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9. Find the period of $2\sin^3 x + 2\cos^3 x$. [2]

$n = 3$ is odd.

From the table for $a\sin^n x + b\cos^n x$, the period is 2π.

Ans: 2π

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10. Find the period of $2|\sin 2x| + 6|\cos 2x|$. [3]

$a|\sin x| + b|\cos x|$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are unequal, so with argument x the period would be π.

The argument is $2x$, so divide by 2:

$\text{Period} = \frac{\pi}{2}$

Ans: $\frac{\pi}{2}$

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11. Find the period of $\frac{5x}{2} - \left[\frac{5x}{2}\right]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{\frac{5}{2}} = \frac{2}{5}$

Ans: $\frac{2}{5}$

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12. Find the period of $f(x) = 2x - [2x] + \sec 5\pi x$. [3]

$\text{Period of } 2x - [2x] = \frac{1}{2}$

$\text{Period of } \sec 5\pi x = \frac{2\pi}{5\pi} = \frac{2}{5}$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{2}, \frac{2}{5}\right\} = \frac{\text{L.C.M.}\{1,2\}}{\text{G.C.D.}\{2,5\}} = 2$

Ans: 2

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13. What is the range of $\cos^3 x$? [1]

From the table of ranges, $\cos^3 x$ takes the values $[-1, 1]$.

Ans: $[-1, 1]$

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14. Find the range of $4 + 2\tan^2 x$. [2]

The range of $\tan^2 x$ is $[0, \infty)$.

$0 \le \tan^2 x < \infty$

Multiply by 2 and add 4.

Range $= [4, \infty)$.

Ans: $[4, \infty)$

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15. Find the maximum and minimum values of $8\sin x + 6\cos x - 2$. [2]

$a = 6,\ b = 8,\ c = -2$

$\sqrt{a^2 + b^2} = \sqrt{36 + 64} = 10$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 8, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -12$

Ans: Maximum 8**, minimum** -12

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16. Find the maximum and minimum values of $f(x) = -3\sin\left(x - \frac{\pi}{6}\right) - 8\cos x + 5$. [5]

Expand with $\sin(x \pm \beta) = \sin x \cos\beta \pm \cos x \sin\beta$:

$f(x) = -\frac{3\sqrt{3}}{2}\sin x + \frac{3}{2}\cos x - 8\cos x + 5$

$= \left(-\frac{13}{2}\right)\cos x + \left(-\frac{3\sqrt{3}}{2}\right)\sin x + 5$

$a^2 + b^2 = \frac{169}{4} + \frac{27}{4} = 49$, so $\sqrt{a^2 + b^2} = 7$

$\text{Maximum} = 5 + 7 = 12, \quad \text{minimum} = 5 - 7 = -2$

Ans: Maximum 12**, minimum** -2

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17. Find the range of $\dfrac{1}{6\cos x + 8\sin x + 2}$. [3]

$\sqrt{36 + 64} = 10$

The denominator has range $[-8, 12]$.

The range of f contains 0 inside it, so the range of $\frac{1}{f}$ is $\left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$.

Range $= (-\infty, -\frac{1}{8}] \cup [\frac{1}{12}, \infty)$.

Ans: $(-\infty, -\frac{1}{8}] \cup [\frac{1}{12}, \infty)$

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18. Find the period and the range of $f(x) = \dfrac{1}{7\cos 6x - 24\sin 6x + 28}$. [5]

The denominator $g(x) = 7\cos 6x - 24\sin 6x + 28$ has period $\frac{2\pi}{6} = \frac{\pi}{3}$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{49 + 576} = 25$

Range of g: $[28 - 25, 28 + 25] = [3, 53]$.

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range of f: $[\frac{1}{53}, \frac{1}{3}]$.

Ans: Period $\frac{\pi}{3}$**; range** $[\frac{1}{53}, \frac{1}{3}]$

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19. Find the minimum value of $4\cot\theta + 9\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$4\cot\theta + 9\tan\theta \ge 2\sqrt{4\cot\theta \cdot 9\tan\theta} = 2\sqrt{36}$

since $\tan\theta\cot\theta = 1$. Equality holds when $4\cot\theta = 9\tan\theta$, which is possible.

Minimum $= 2\sqrt{36} = 12$.

Ans: 12

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20. Using A.M. $\geq$ G.M., find the minimum value of $25\cos^2\theta + 49\sec^2\theta$, and say where it is reached. [5]

$25\cos^2\theta + 49\sec^2\theta = (5)^2\cos^2\theta + (7)^2\sec^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $25\cos^2\theta + 49\sec^2\theta \geq 2\sqrt{25 \cdot 49} = 70$

Equality needs $25\cos^2\theta = 49\sec^2\theta$, that is $\cos^2\theta = \frac{7}{5}$.

But $\frac{7}{5} > 1$, so equality is impossible; the bound 70 is not reached.

With $u = \cos^2\theta \in (0, 1]$, the value $25u + \frac{49}{u}$ decreases as u increases up to 1.

Minimum at $\cos^2\theta = 1$: $25 + 49 = 74$.

Ans: Minimum 74, at $\cos^2\theta = 1$