EXPRESSWALA ACADEMY

Maths 1A: Periodicity and Extreme Values

Question Paper, Set 01

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. f is a periodic function with period 9, and $f(-2) = 8$. Find $f(-11)$. [1 mark]

2. Write down the periods of $|\sec x|$ and $\sec x$. [1 mark]

3. Find the period of $\cot(3x + 3)$. [1 mark]

4. Find the period of $7\,\mathrm{cosec}\left(\frac{x}{4} + 1\right) - 1$. [2 marks]

5. Find the period of $\sin 3x + \cot 2x$. [2 marks]

6. Find the period of $f(x) = \sec\frac{x}{4} - \tan\frac{x}{2} - \cos\frac{3x}{5}$. [5 marks]

7. Find the period of $\cos^7 x$. [1 mark]

8. Find the period of $\sin^6\frac{x}{2}$. [2 marks]

9. Find the period of $2\tan^2 x + 2\cot^2 x$. [2 marks]

10. Find the period of $2\tan^2\frac{x}{2} + 2\cot^2\frac{x}{2}$. [3 marks]

11. Find the period of $\frac{2x}{3} - \left[\frac{2x}{3}\right]$. [1 mark]

12. Find the period of $f(x) = 4x - [4x] + \mathrm{cosec}\,\frac{\pi x}{3}$. [3 marks]

13. What is the range of $\sin x$? [1 mark]

14. Find the range of $-1 + 4\sin^4 x$. [2 marks]

15. Find the maximum and minimum values of $5\cos x + 12\sin x + 3$. [2 marks]

16. Find the maximum and minimum values of $f(x) = 8\sin\left(x + \frac{\pi}{6}\right) - 3\cos x + 2$. [5 marks]

17. Find the range of $\dfrac{1}{-3 - 2\sin x}$. [3 marks]

18. Find the period and the range of $f(x) = \dfrac{1}{7\cos\frac{x}{3} + 24\sin\frac{x}{3} + 26}$. [5 marks]

19. Find the minimum value of $3\tan\theta + 12\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3 marks]

20. Using A.M. $\geq$ G.M., find the minimum value of $16\sin^2\theta + 1\,\mathrm{cosec}^2\,\theta$, and say where it is reached. [5 marks]

Key sheet

Q	Answer
1	8
2	π and 2π
3	$\frac{\pi}{3}$
4	8π
5	2π
6	40π
7	2π
8	2π
9	$\frac{\pi}{2}$
10	π
11	$\frac{3}{2}$
12	6
13	$[-1, 1]$
14	$[-1, 3]$
15	Maximum 16, minimum -10
16	Maximum 9, minimum -5
17	$[-1, -\frac{1}{5}]$
18	Period 6π; range $[\frac{1}{51}, 1]$
19	12
20	Minimum 8, at $\sin^2\theta = \frac{1}{4}$

Solutions

1. f is a periodic function with period 9, and $f(-2) = 8$. Find $f(-11)$. [1]

$-11 = -2 - 1 \times 9$

$f(x + np) = f(x) \Rightarrow f(-11) = f(-2) = 8$

Ans: 8

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2. Write down the periods of $|\sec x|$ and $\sec x$. [1]

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ have period 2π; $\tan x, \cot x$ and the absolute value of any of the six have period π.

So the periods are π and 2π.

Ans: π **and** 2π

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3. Find the period of $\cot(3x + 3)$. [1]

The period of $\cot x$ is π; for $f(ax + b)$ the period is $\frac{p}{|a|}$.

$\text{Period} = \frac{\pi}{3}$

Ans: $\frac{\pi}{3}$

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4. Find the period of $7\operatorname{cosec}\left(\frac{x}{4} + 1\right) - 1$. [2]

The period of $\operatorname{cosec} x$ is 2π, and here $a = \frac{1}{4}$.

The constants 7 and -1 do not change the period.

$\text{Period} = \frac{2\pi}{\frac{1}{4}} = 8\pi$

Ans: 8π

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5. Find the period of $\sin 3x + \cot 2x$. [2]

$\text{Period of } \sin 3x = \frac{2\pi}{3}, \quad \text{period of } \cot 2x = \frac{\pi}{2}$

$\text{L.C.M.}\left\{\frac{2\pi}{3}, \frac{\pi}{2}\right\} = 2\pi$

Period $= 2\pi$.

Ans: 2π

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6. Find the period of $f(x) = \sec\frac{x}{4} - \tan\frac{x}{2} - \cos\frac{3x}{5}$. [5]

$\text{Period of } \sec\frac{x}{4} = \frac{2\pi}{\frac{1}{4}} = 8\pi$

$\text{Period of } \tan\frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$

$\text{Period of } \cos\frac{3x}{5} = \frac{2\pi}{\frac{3}{5}} = \frac{10\pi}{3}$

Write each period as a fraction times π: $\frac{8}{1}\pi$, $\frac{2}{1}\pi$, $\frac{10}{3}\pi$.

$\text{L.C.M.} = \frac{\text{L.C.M.}\{8,2,10\}}{\text{G.C.D.}\{1,1,3\}}\,\pi = \frac{40}{1}\,\pi$

No two terms swap into each other, so the period of f is the L.C.M.: 40π.

Ans: 40π

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7. Find the period of $\cos^7 x$. [1]

$\cos^n x$ has period π when n is even and 2π when n is odd; here $n = 7$ is odd.

Period $= 2\pi$.

Ans: 2π

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8. Find the period of $\sin^6 \frac{x}{2}$. [2]

$\sin^6 x$ has period π (6 is even).

$\text{Period of } \sin^6 \frac{x}{2} = \frac{\pi}{\frac{1}{2}} = 2\pi$

Ans: 2π

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9. Find the period of $2\tan^2 x + 2\cot^2 x$. [2]

$n = 2$ is even and the coefficients are equal.

Shifting x by $\frac{\pi}{2}$ swaps the two terms, so the sum repeats after $\frac{\pi}{2}$.

From the table for $a\tan^n x + b\cot^n x$, the period is $\frac{\pi}{2}$.

Ans: $\frac{\pi}{2}$

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10. Find the period of $2\tan^2 \frac{x}{2} + 2\cot^2 \frac{x}{2}$. [3]

For even n, $a\tan^n x + b\cot^n x$ has period $\frac{\pi}{2}$ if $a = b$ and π if $a \neq b$.

Here the coefficients are equal, so with argument x the period would be $\frac{\pi}{2}$.

The argument is $\frac{x}{2}$, so divide by $\frac{1}{2}$:

$\text{Period} = \frac{\frac{\pi}{2}}{\frac{1}{2}} = \pi$

Ans: π

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11. Find the period of $\frac{2x}{3} - \left[\frac{2x}{3}\right]$. [1]

$x - [x]$ has period 1, so $ax - [ax]$ has period $\frac{1}{|a|}$.

$\text{Period} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

Ans: $\frac{3}{2}$

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12. Find the period of $f(x) = 4x - [4x] + \operatorname{cosec} \frac{\pi x}{3}$. [3]

$\text{Period of } 4x - [4x] = \frac{1}{4}$

$\text{Period of } \operatorname{cosec} \frac{\pi x}{3} = \frac{2\pi}{\frac{\pi}{3}} = 6$

Both periods are rational, so the L.C.M. exists:

$\text{L.C.M.}\left\{\frac{1}{4}, 6\right\} = \frac{\text{L.C.M.}\{1,6\}}{\text{G.C.D.}\{4,1\}} = 6$

Ans: 6

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13. What is the range of $\sin x$? [1]

From the table of ranges, $\sin x$ takes the values $[-1, 1]$.

Ans: $[-1, 1]$

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14. Find the range of $-1 + 4\sin^4 x$. [2]

The range of $\sin^4 x$ is $[0, 1]$.

$0 \le \sin^4 x \le 1$

Multiply by 4 and add -1.

Range $= [-1, 3]$.

Ans: $[-1, 3]$

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15. Find the maximum and minimum values of $5\cos x + 12\sin x + 3$. [2]

$a = 5,\ b = 12,\ c = 3$

$\sqrt{a^2 + b^2} = \sqrt{25 + 144} = 13$

$\text{Maximum} = c + \sqrt{a^2 + b^2} = 16, \quad \text{minimum} = c - \sqrt{a^2 + b^2} = -10$

Ans: Maximum 16**, minimum** -10

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16. Find the maximum and minimum values of $f(x) = 8\sin\left(x + \frac{\pi}{6}\right) - 3\cos x + 2$. [5]

Expand with $\sin(x \pm \beta) = \sin x \cos\beta \pm \cos x \sin\beta$:

$f(x) = 4\sqrt{3}\sin x + 4\cos x - 3\cos x + 2$

$= (1)\cos x + \left(4\sqrt{3}\right)\sin x + 2$

$a^2 + b^2 = 1 + 48 = 49$, so $\sqrt{a^2 + b^2} = 7$

$\text{Maximum} = 2 + 7 = 9, \quad \text{minimum} = 2 - 7 = -5$

Ans: Maximum 9**, minimum** -5

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17. Find the range of $\dfrac{1}{-3 - 2\sin x}$. [3]

$-1 \le \sin x \le 1 \Rightarrow -5 \le -3 - 2\sin x \le -1$

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range $= [-1, -\frac{1}{5}]$.

Ans: $[-1, -\frac{1}{5}]$

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18. Find the period and the range of $f(x) = \dfrac{1}{7\cos\frac{x}{3} + 24\sin\frac{x}{3} + 26}$. [5]

The denominator $g(x) = 7\cos\frac{x}{3} + 24\sin\frac{x}{3} + 26$ has period $\frac{2\pi}{\frac{1}{3}} = 6\pi$; so does $f = \frac{1}{g}$.

$\sqrt{a^2 + b^2} = \sqrt{49 + 576} = 25$

Range of g: $[26 - 25, 26 + 25] = [1, 51]$.

Both ends have the same sign, so the range of $\frac{1}{f}$ is $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Range of f: $[\frac{1}{51}, 1]$.

Ans: Period 6π**; range** $[\frac{1}{51}, 1]$

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19. Find the minimum value of $3\tan\theta + 12\cot\theta$ for $0 < \theta < \frac{\pi}{2}$. [3]

For acute θ both terms are positive. By A.M. $\ge$ G.M.:

$3\tan\theta + 12\cot\theta \ge 2\sqrt{3\tan\theta \cdot 12\cot\theta} = 2\sqrt{36}$

since $\tan\theta\cot\theta = 1$. Equality holds when $3\tan\theta = 12\cot\theta$, which is possible.

Minimum $= 2\sqrt{36} = 12$.

Ans: 12

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20. Using A.M. $\geq$ G.M., find the minimum value of $16\sin^2\theta + 1\,\mathrm{cosec}^2\theta$, and say where it is reached. [5]

$16\sin^2\theta + 1\,\mathrm{cosec}^2\theta = (4)^2\sin^2\theta + (1)^2\,\mathrm{cosec}^2\theta$

By A.M. $\geq$ G.M. $u + v \geq 2\sqrt{uv}$: $16\sin^2\theta + 1\,\mathrm{cosec}^2\theta \geq 2\sqrt{16 \cdot 1} = 8$

Equality needs $16\sin^2\theta = 1\,\mathrm{cosec}^2\theta$, that is $\sin^2\theta = \frac{1}{4}$.

Since $\frac{1}{4} \leq 1$, this is possible.

Minimum $= 8$, reached when $\sin^2\theta = \frac{1}{4}$.

Ans: Minimum 8**, at** $\sin^2\theta = \frac{1}{4}$