EXPRESSWALA ACADEMY

Maths 1A: Multiple and Submultiple Angles

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\tan A = -\frac{5}{7}$, find $\sin 2A$. [1 mark]

2. If $\cos A = \frac{4}{5}$ and $\frac{3\pi}{2} < A < 2\pi$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$. [3 marks]

3. Using $\cos 2A$, find the exact value of $\sin^2 105^\circ$. [2 marks]

4. If $\cos A = -\frac{1}{6}$, find $\cos 3A$. [2 marks]

5. If $\cos A = \frac{3}{5}$ and $\pi < A < 2\pi$, find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$ and $\tan \frac{A}{2}$. [3 marks]

6. If $\tan \frac{A}{2} = \frac{7}{4}$, find $\tan A$. [1 mark]

7. Show that $\cos 3A = 4\cos^3 A - 3\cos A$. [5 marks]

8. Write the exact value of $\cos 54^\circ$. [1 mark]

9. Show that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$. Hence find $\sin 18^\circ \sin 54^\circ$. [5 marks]

10. Find the exact value of $\tan^2 52\frac{1}{2}^\circ + \cot^2 52\frac{1}{2}^\circ$. [2 marks]

11. If $\sin 2\theta = -\frac{6}{7}$, find $\sin^2\theta + \cos^4\theta$. [1 mark]

12. Find the value of $\tan 52\frac{1}{2}^\circ - \tan 37\frac{1}{2}^\circ$. [2 marks]

13. Prove that $\tan A + 2\tan 2A + 4\cot 4A = \cot A$. [5 marks]

14. Find the value of $\tan \frac{\pi}{24}\left(1 + \sec \frac{\pi}{12}\right)\left(1 + \sec \frac{\pi}{6}\right)$. [3 marks]

15. Find the value of $\cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7}$. [3 marks]

16. Find the value of $\cos \frac{\pi}{7} \cos \frac{3\pi}{7} \cos \frac{5\pi}{7} + \frac{1}{8}$. [1 mark]

17. Prove that $\cos\theta \cos(60^\circ - \theta)\cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$ and hence find $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ$. [5 marks]

18. If $\frac{x}{y} = \frac{\tan 75^\circ}{\tan 15^\circ}$, find $\frac{x+y}{x-y}$. [2 marks]

19. For $A = 520^\circ$, is $\cos \frac{A}{2} - \sin \frac{A}{2}$ positive or negative? [1 mark]

20. Simplify $\sqrt{1-\sin A}$ for $A = 210^\circ$ and find its value. [2 marks]

Key sheet

Q	Answer
1	$-\frac{35}{37}$
2	$\sin 2A = -\frac{24}{25},\ \cos 2A = \frac{7}{25},\ \tan 2A = -\frac{24}{7}$
3	$\frac{2+\sqrt{3}}{4}$
4	$\frac{13}{27}$
5	$\sin\frac{A}{2} = \frac{\sqrt{5}}{5},\ \cos\frac{A}{2} = -\frac{2\sqrt{5}}{5},\ \tan\frac{A}{2} = -\frac{1}{2}$
6	$-\frac{56}{33}$
7	Proved: $\cos 3A = 4\cos^3 A - 3\cos A$
8	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
9	$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$; $\sin 18^\circ \sin 54^\circ = \frac{1}{4}$
10	$30 - 16\sqrt{3}$
11	$\frac{40}{49}$
12	$4 - 2\sqrt{3}$
13	Proved: $\tan A + 2\tan 2A + 4\cot 4A = \cot A$
14	$\frac{\sqrt{3}}{3}$
15	$\frac{1}{2}$
16	0
17	Proved; $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$
18	$\frac{2\sqrt{3}}{3}$
19	Positive
20	$-(\cos 105^\circ - \sin 105^\circ) = \frac{\sqrt{6}}{2}$

Solutions

1. If $\tan A = -\frac{5}{7}$, find $\sin 2A$. [1]

$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

$= -\frac{35}{37}$

Ans: $-\frac{35}{37}$

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2. If $\cos A = \frac{4}{5}$ and $\frac{3\pi}{2} < A < 2\pi$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$. [3]

$\sin A = -\sqrt{1-\cos^2 A} = -\frac{3}{5}$ ($\sin A$ is negative in this quadrant)

$\sin 2A = 2\sin A \cos A = 2(-\frac{3}{5})(\frac{4}{5}) = -\frac{24}{25}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

$\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{24}{7}$

Ans: $\sin 2A = -\frac{24}{25},\ \cos 2A = \frac{7}{25},\ \tan 2A = -\frac{24}{7}$

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3. Using $\cos 2A$, find the exact value of $\sin^2 105°$. [2]

$\sin^2 A = \frac{1-\cos 2A}{2}$

$\sin^2 105° = \frac{1-\cos 210°}{2}$

$\cos 210° = -\frac{\sqrt{3}}{2}$

$= \frac{2+\sqrt{3}}{4}$

Ans: $\frac{2+\sqrt{3}}{4}$

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4. If $\cos A = -\frac{1}{6}$, find $\cos 3A$. [2]

$\cos 3A = 4\cos^3 A - 3\cos A$

$= 4(-\frac{1}{6})^3 - 3(-\frac{1}{6})$

$= \frac{13}{27}$

Ans: $\frac{13}{27}$

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5. If $\cos A = \frac{3}{5}$ and $\pi < A < 2\pi$, find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$. [3]

Here $\frac{\pi}{2} < \frac{A}{2} < \pi$, so $\sin\frac{A}{2}$ is positive and $\cos\frac{A}{2}$ is negative.

$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \frac{\sqrt{5}}{5}$

$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\frac{2\sqrt{5}}{5}$

$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -\frac{1}{2}$

Ans: $\sin\frac{A}{2} = \frac{\sqrt{5}}{5},\ \cos\frac{A}{2} = -\frac{2\sqrt{5}}{5},\ \tan\frac{A}{2} = -\frac{1}{2}$

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6. If $\tan\frac{A}{2} = \frac{7}{4}$, find $\tan A$. [1]

$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$= -\frac{56}{33}$

Ans: $-\frac{56}{33}$

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7. Show that $\cos 3A = 4\cos^3 A - 3\cos A$. [5]

$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 4\cos^3 A - 3\cos A$

Ans: Proved: $\cos 3A = 4\cos^3 A - 3\cos A$

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8. Write the exact value of $\cos 54°$. [1]

$\cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4}$

Ans: $\frac{\sqrt{10-2\sqrt{5}}}{4}$

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9. Show that $\sin 18° = \frac{\sqrt{5}-1}{4}$. Hence find $\sin 18° \sin 54°$. [5]

Let $\theta = 18°$. Then $5\theta = 90°$, so $2\theta = 90° - 3\theta$ and $\sin 2\theta = \cos 3\theta$.

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$; divide by $\cos\theta \neq 0$: $2\sin\theta = 4(1 - \sin^2\theta) - 3$

$4\sin^2\theta + 2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-1\pm\sqrt{5}}{4}$

$18°$ is acute, so $\sin 18° > 0$ and $\sin 18° = \frac{\sqrt{5}-1}{4}$.

$\sin 54° = \cos 36° = \frac{\sqrt{5}+1}{4}$

Product $= \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16} = \frac{1}{4}$

Ans: $\sin 18° = \frac{\sqrt{5}-1}{4}$; $\sin 18° \sin 54° = \frac{1}{4}$

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10. Find the exact value of $\tan^2 52\frac{1}{2}^\circ + \cot^2 52\frac{1}{2}^\circ$. [2]

$\tan 52\frac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}-1),\ \cot 52\frac{1}{2}^\circ = (\sqrt{3}-\sqrt{2})(\sqrt{2}+1)$

$\tan^2 52\frac{1}{2}^\circ + \cot^2 52\frac{1}{2}^\circ = 30 - 16\sqrt{3}$

Ans: $30 - 16\sqrt{3}$

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11. If $\sin 2\theta = -\frac{6}{7}$, find $\sin^2\theta + \cos^4\theta$. [1]

$\sin^2\theta + \cos^4\theta = 1 - \frac{1}{4}\sin^2 2\theta$

$= 1 - \frac{1}{4} \cdot \frac{36}{49} = \frac{40}{49}$

Ans: $\frac{40}{49}$

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12. Find the value of $\tan 52\frac{1}{2}^\circ - \tan 37\frac{1}{2}^\circ$. [2]

$\tan(45° + A) - \tan(45° - A) = 2\tan 2A$

With $A = 7\frac{1}{2}^\circ$: $\tan 52\frac{1}{2}^\circ - \tan 37\frac{1}{2}^\circ = 4 - 2\sqrt{3}$

Ans: $4 - 2\sqrt{3}$

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13. Prove that $\tan A + 2\tan 2A + 4\cot 4A = \cot A$. [5]

From $\cot\theta - \tan\theta = 2\cot 2\theta$: $\tan\theta = \cot\theta - 2\cot 2\theta$

$\tan A = \cot A - 2\cot 2A$

$2\tan 2A = 2\cot 2A - 4\cot 4A$

Adding: $\tan A + 2\tan 2A = \cot A - 4\cot 4A$, so $\tan A + 2\tan 2A + 4\cot 4A = \cot A$

Ans: Proved: $\tan A + 2\tan 2A + 4\cot 4A = \cot A$

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14. Find the value of $\tan\frac{\pi}{24}\left(1+\sec\frac{\pi}{12}\right)\left(1+\sec\frac{\pi}{6}\right)$. [3]

$\tan\frac{\theta}{2}(1+\sec\theta)=\tan\theta$, and repeating, the product is $\tan 2\theta$ with $\theta=\frac{\pi}{12}$.

$\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3}$

Ans: $\frac{\sqrt{3}}{3}$

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15. Find the value of $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}$. [3]

$a_1=\frac{\pi}{7}$, $d=\frac{2\pi}{7}$, $n=3$, $a_n=\frac{5\pi}{7}$

$\text{Sum}=\frac{\cos\left(\frac{a_1+a_n}{2}\right)\sin\frac{nd}{2}}{\sin\frac{d}{2}}=\frac{\cos\frac{3\pi}{7}\ \sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$

The angles in the numerator add to π, so $\cos x\sin x=\frac{1}{2}\sin 2x$ and $\sin(\pi-y)=\sin y$ cancel the denominator.

$=\frac{1}{2}$

Ans: $\frac{1}{2}$

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16. Find the value of $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}+\frac{1}{8}$. [1]

Known result: $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\frac{1}{8}$

$\text{Value}=0$

Ans: 0

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17. Prove that $\cos\theta\cos(60^\circ-\theta)\cos(60^\circ+\theta)=\frac{1}{4}\cos 3\theta$ and hence find $\cos 10^\circ\cos 30^\circ\cos 50^\circ\cos 70^\circ$. [5]

$\cos(60^\circ+\theta)\cos(60^\circ-\theta)=\cos^2 60^\circ-\sin^2\theta=\cos^2\theta-\frac{3}{4}$

LHS $=\cos\theta\left(\cos^2\theta-\frac{3}{4}\right)=\frac{1}{4}(4\cos^3\theta-3\cos\theta)$

$=\frac{1}{4}\cos 3\theta$ = RHS

With $\theta=10^\circ$: $\cos 10^\circ\cos 50^\circ\cos 70^\circ=\frac{\sqrt{3}}{8}$

Multiply by $\cos 30^\circ=\frac{\sqrt{3}}{2}$: the value is $\frac{3}{16}$.

Ans: Proved; $\cos 10^\circ\cos 30^\circ\cos 50^\circ\cos 70^\circ=\frac{3}{16}$

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18. If $\frac{x}{y}=\frac{\tan 75^\circ}{\tan 15^\circ}$, find $\frac{x+y}{x-y}$. [2]

$\frac{x+y}{x-y}=\frac{\tan A+\tan B}{\tan A-\tan B}=\frac{\sin(A+B)}{\sin(A-B)}$

$=\frac{\sin 90^\circ}{\sin 60^\circ}=\frac{2\sqrt{3}}{3}$

Ans: $\frac{2\sqrt{3}}{3}$

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19. For $A=520^\circ$, is $\cos\frac{A}{2}-\sin\frac{A}{2}$ positive or negative? [1]

$\frac{A}{2}=260^\circ$

$c-s>0$ for $\frac{A}{2}$ in $\left(-\frac{3\pi}{4},\frac{\pi}{4}\right)$ and <0 in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$ (adding 2π where needed).

So $\cos 260^\circ-\sin 260^\circ$ is positive.

Ans: Positive

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20. Simplify $\sqrt{1-\sin A}$ for $A=210^\circ$ and find its value. [2]

$\sqrt{1-\sin A}=\left|\cos\frac{A}{2}-\sin\frac{A}{2}\right|$

$\frac{A}{2}=105^\circ$, where $\cos\frac{A}{2}-\sin\frac{A}{2}$ is negative.

$\sqrt{1-\sin 210^\circ}=-(\cos 105^\circ-\sin 105^\circ)=\frac{\sqrt{6}}{2}$

Ans: $-(\cos 105^\circ-\sin 105^\circ)=\frac{\sqrt{6}}{2}$
