EXPRESSWALA ACADEMY

Maths 1A: Multiple and Submultiple Angles

Question Paper, Set 02

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\tan A = \frac{7}{3}$, find $\tan 2A$. [1 mark]

2. If $\cos A = \frac{3}{5}$ and $\frac{3\pi}{2} < A < 2\pi$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$. [3 marks]

3. If $\cos 2A = \frac{2}{9}$, find $\cos^2 A$. [2 marks]

4. If $\tan A = \frac{1}{3}$, find $\tan 3A$. [2 marks]

5. If $\cos A = \frac{1}{2}$ and $0 < A < \pi$, find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$ and $\tan \frac{A}{2}$. [3 marks]

6. If $\tan \frac{A}{2} = \frac{8}{5}$, find $\cos A$. [1 mark]

7. Show that $\frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4\cos 2A$. [5 marks]

8. Write the exact value of $\cos 198^\circ$. [1 mark]

9. Show that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$. Hence find $\cos 18^\circ$. [5 marks]

10. Find the exact value of $\tan 82\frac{1}{2}^\circ \tan 7\frac{1}{2}^\circ$. [2 marks]

11. If $\sin 2\theta = \frac{1}{7}$, find $\sin^6\theta + \cos^6\theta$. [1 mark]

12. Find the value of $\tan 82\frac{1}{2}^\circ - \tan 7\frac{1}{2}^\circ$. [2 marks]

13. Prove that $\tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = 2\tan 2A$. [5 marks]

14. Find the value of $(2\cos 7\frac{1}{2}^\circ - 1)(2\cos 7\frac{1}{2}^\circ + 1)(2\cos 15^\circ - 1)$. [3 marks]

15. Find the value of $\sin 36^\circ + \sin 72^\circ + \sin 108^\circ + \sin 144^\circ$. [3 marks]

16. Find the value of $\left(\tan\frac{\pi}{7} + \tan\frac{2\pi}{7} + \tan\frac{4\pi}{7}\right)^2$. [1 mark]

17. Prove that $\tan\theta\tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan 3\theta$ and hence find $\tan 5^\circ \tan 55^\circ \tan 65^\circ$. [5 marks]

18. If $\frac{x}{y} = \frac{\tan 97\frac{1}{2}^\circ}{\tan 52\frac{1}{2}^\circ}$, find $\frac{x+y}{x-y}$. [2 marks]

19. For $A = 200^\circ$, is $\cos\frac{A}{2} - \sin\frac{A}{2}$ positive or negative? [1 mark]

20. Simplify $\sqrt{1 + \sin A}$ for $A = 570^\circ$ and find its value. [2 marks]

Key sheet

Q	Answer
1	$-\frac{21}{20}$
2	$\sin 2A = -\frac{24}{25},\ \cos 2A = -\frac{7}{25},\ \tan 2A = \frac{24}{7}$
3	$\frac{11}{18}$
4	$\frac{13}{9}$
5	$\sin\frac{A}{2} = \frac{1}{2},\ \cos\frac{A}{2} = \frac{\sqrt{3}}{2},\ \tan\frac{A}{2} = \frac{\sqrt{3}}{3}$
6	$-\frac{39}{89}$
7	Proved: $\frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4\cos 2A$
8	$-\frac{\sqrt{10+2\sqrt{5}}}{4}$
9	$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$; $\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$
10	1
11	$\frac{193}{196}$
12	$4 + 2\sqrt{3}$
13	Proved: $\tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = 2\tan 2A$
14	$1 + \sqrt{3}$
15	$\cot 18^\circ$
16	7
17	Proved; $\tan 5^\circ \tan 55^\circ \tan 65^\circ = 2 - \sqrt{3}$
18	$\frac{\sqrt{2}}{2}$
19	Negative
20	$-(\cos 285^\circ + \sin 285^\circ) = \frac{\sqrt{2}}{2}$

Solutions

1. If $\tan A = \frac{7}{3}$, find $\tan 2A$. [1]

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$= -\frac{21}{20}$

Ans: $-\frac{21}{20}$

---

2. If $\cos A = \frac{3}{5}$ and $\frac{3\pi}{2} < A < 2\pi$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$. [3]

$\sin A = -\sqrt{1-\cos^2 A} = -\frac{4}{5}$ ($\sin A$ is negative in this quadrant)

$\sin 2A = 2\sin A \cos A = 2(-\frac{4}{5})(\frac{3}{5}) = -\frac{24}{25}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$

$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{7}$

Ans: $\sin 2A = -\frac{24}{25},\ \cos 2A = -\frac{7}{25},\ \tan 2A = \frac{24}{7}$

---

3. If $\cos 2A = \frac{2}{9}$, find $\cos^2 A$. [2]

$\cos^2 A = \frac{1+\cos 2A}{2}$

$= \frac{11}{18}$

Ans: $\frac{11}{18}$

---

4. If $\tan A = \frac{1}{3}$, find $\tan 3A$. [2]

$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

$= \frac{3(\frac{1}{3}) - (\frac{1}{3})^3}{1-3(\frac{1}{3})^2}$

$= \frac{13}{9}$

Ans: $\frac{13}{9}$

---

5. If $\cos A = \frac{1}{2}$ and $0 < A < \pi$, find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$. [3]

Here $0 < \frac{A}{2} < \frac{\pi}{2}$, so $\sin\frac{A}{2}$ is positive and $\cos\frac{A}{2}$ is positive.

$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \frac{1}{2}$

$\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}} = \frac{\sqrt{3}}{2}$

$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{3}}{3}$

Ans: $\sin\frac{A}{2} = \frac{1}{2},\ \cos\frac{A}{2} = \frac{\sqrt{3}}{2},\ \tan\frac{A}{2} = \frac{\sqrt{3}}{3}$

---

6. If $\tan\frac{A}{2} = \frac{8}{5}$, find $\cos A$. [1]

$\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

$= -\frac{39}{89}$

Ans: $-\frac{39}{89}$

---

7. Show that $\frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4\cos 2A$. [5]

$\frac{\sin 3A}{\sin A} = \frac{3\sin A - 4\sin^3 A}{\sin A} = 3 - 4\sin^2 A$

$\frac{\cos 3A}{\cos A} = \frac{4\cos^3 A - 3\cos A}{\cos A} = 4\cos^2 A - 3$

LHS $= 4\cos^2 A - 4\sin^2 A = 4(\cos^2 A - \sin^2 A)$

$= 4\cos 2A$ = RHS

Ans: Proved: $\frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4\cos 2A$

---

8. Write the exact value of $\cos 198°$. [1]

$\cos 198° = -\cos 18°$

$\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$

Ans: $-\frac{\sqrt{10+2\sqrt{5}}}{4}$

---

9. Show that $\sin 18° = \frac{\sqrt{5}-1}{4}$. Hence find $\cos 18°$. [5]

Let $\theta = 18°$. Then $5\theta = 90°$, so $2\theta = 90° - 3\theta$ and $\sin 2\theta = \cos 3\theta$.

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$; divide by $\cos\theta \neq 0$: $2\sin\theta = 4(1 - \sin^2\theta) - 3$

$4\sin^2\theta + 2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-1\pm\sqrt{5}}{4}$

$18°$ is acute, so $\sin 18° > 0$ and $\sin 18° = \frac{\sqrt{5}-1}{4}$.

$\cos^2 18° = 1 - \frac{6-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$

$\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$ (positive, acute angle)

Ans: $\sin 18° = \frac{\sqrt{5}-1}{4}$; $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$

---

10. Find the exact value of $\tan 82\frac{1}{2}° \tan 7\frac{1}{2}°$. [2]

$\tan 82\frac{1}{2}° = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$, $\tan 7\frac{1}{2}° = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$

$\tan 82\frac{1}{2}° \tan 7\frac{1}{2}° = 1$

Ans: 1

---

11. If $\sin 2\theta = \frac{1}{7}$, find $\sin^6\theta + \cos^6\theta$. [1]

$\sin^6\theta + \cos^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$

$= 1 - \frac{3}{4}\cdot\frac{1}{49} = \frac{193}{196}$

Ans: $\frac{193}{196}$

---

12. Find the value of $\tan 82\frac{1}{2}° - \tan 7\frac{1}{2}°$. [2]

$\tan(45° + A) - \tan(45° - A) = 2\tan 2A$

With $A = 37\frac{1}{2}°$: $\tan 82\frac{1}{2}° - \tan 7\frac{1}{2}° = 4 + 2\sqrt{3}$

Ans: $4 + 2\sqrt{3}$

---

13. Prove that $\tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = 2\tan 2A$. [5]

Let $t = \tan A$: LHS $= \frac{1+t}{1-t} - \frac{1-t}{1+t}$

$= \frac{(1+t)^2 - (1-t)^2}{1-t^2} = \frac{4t}{1-t^2}$

$= 2\cdot\frac{2t}{1-t^2} = 2\tan 2A$ = RHS

Ans: Proved: $\tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = 2\tan 2A$

---

14. Find the value of $(2\cos 7\frac{1}{2}^\circ - 1)(2\cos 7\frac{1}{2}^\circ + 1)(2\cos 15^\circ - 1)$. [3]

$(2\cos\theta - 1)(2\cos\theta + 1) = 2\cos 2\theta + 1$, then $(2\cos 2\theta + 1)(2\cos 2\theta - 1) = 2\cos 4\theta + 1$

$= 2\cos 30^\circ + 1 = 1 + \sqrt{3}$

Ans: $1 + \sqrt{3}$

---

15. Find the value of $\sin 36^\circ + \sin 72^\circ + \sin 108^\circ + \sin 144^\circ$. [3]

$a_1 = 36^\circ$, $d = 36^\circ$, $n = 4$, $a_n = 144^\circ$

$\text{Sum} = \frac{\sin\left(\frac{a_1+a_n}{2}\right)\sin\frac{nd}{2}}{\sin\frac{d}{2}} = \frac{\sin 90^\circ \sin 72^\circ}{\sin 18^\circ}$

$= \cot 18^\circ$

Ans: $\cot 18^\circ$

---

16. Find the value of $\left(\tan\frac{\pi}{7} + \tan\frac{2\pi}{7} + \tan\frac{4\pi}{7}\right)^2$. [1]

Known result: $\tan\frac{\pi}{7} + \tan\frac{2\pi}{7} + \tan\frac{4\pi}{7} = -\sqrt{7}$

$\text{Square} = 7$

Ans: 7

---

17. Prove that $\tan\theta\tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan 3\theta$ and hence find $\tan 5^\circ \tan 55^\circ \tan 65^\circ$. [5]

$\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$ (using $\sin^2 60^\circ = \frac{3}{4}$)

$\cos\theta\cos(60^\circ - \theta)\cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$ (using $\cos^2 60^\circ = \frac{1}{4}$)

Dividing: $\tan\theta\tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan 3\theta$

With $\theta = 5^\circ$: $\tan 5^\circ \tan 55^\circ \tan 65^\circ = 2 - \sqrt{3}$

Ans: Proved; $\tan 5^\circ \tan 55^\circ \tan 65^\circ = 2 - \sqrt{3}$

---

18. If $\frac{x}{y} = \frac{\tan 97\frac{1}{2}^\circ}{\tan 52\frac{1}{2}^\circ}$, find $\frac{x+y}{x-y}$. [2]

$\frac{x+y}{x-y} = \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}$

$= \frac{\sin 150^\circ}{\sin 45^\circ} = \frac{\sqrt{2}}{2}$

Ans: $\frac{\sqrt{2}}{2}$

---

19. For $A = 200^\circ$, is $\cos\frac{A}{2} - \sin\frac{A}{2}$ positive or negative? [1]

$\frac{A}{2} = 100^\circ$

$c - s > 0$ for $\frac{A}{2}$ in $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$ and < 0 in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ (adding 2π where needed).

So $\cos 100^\circ - \sin 100^\circ$ is negative.

Ans: Negative

---

20. Simplify $\sqrt{1 + \sin A}$ for $A = 570^\circ$ and find its value. [2]

$\sqrt{1 + \sin A} = \left|\cos\frac{A}{2} + \sin\frac{A}{2}\right|$

$\frac{A}{2} = 285^\circ$, where $\cos\frac{A}{2} + \sin\frac{A}{2}$ is negative.

$\sqrt{1 + \sin 570^\circ} = -(\cos 285^\circ + \sin 285^\circ) = \frac{\sqrt{2}}{2}$

Ans: $-(\cos 285^\circ + \sin 285^\circ) = \frac{\sqrt{2}}{2}$
