EXPRESSWALA ACADEMY

Maths 1A: Multiple and Submultiple Angles

Question Paper, Set 01

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\tan A = -\frac{5}{4}$, find $\tan 2A$. [1 mark]

2. If $\sin A = -\frac{24}{25}$ and $\pi < A < \frac{3\pi}{2}$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$. [3 marks]

3. If $\cos 2A = \frac{1}{2}$ and $\pi < A < \frac{3\pi}{2}$, find $\cos A$. [2 marks]

4. If $\sin A = \frac{2}{7}$, find $\sin 3A$. [2 marks]

5. If $\cos A = \frac{2}{3}$ and $\pi < A < 2\pi$, find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$ and $\tan \frac{A}{2}$. [3 marks]

6. If $\tan \frac{A}{2} = -\frac{3}{7}$, find $\tan A$. [1 mark]

7. Prove that $\frac{1+\sin A-\cos A}{1+\sin A+\cos A} = \tan \frac{A}{2}$. [5 marks]

8. Write the exact value of $\sin 108^\circ$. [1 mark]

9. Show that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$. Hence find $\cos 36^\circ$. [5 marks]

10. Find the exact value of $\tan 37\frac{1}{2}^\circ + \cot 37\frac{1}{2}^\circ$. [2 marks]

11. If $\sin 2\theta = \frac{1}{4}$, find $\sin^2\theta + \cos^4\theta$. [1 mark]

12. Find the value of $\cot 60^\circ - \tan 60^\circ$. [2 marks]

13. Prove that $\tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right) = 2\sec 2A$. [5 marks]

14. Find the value of $\left(2\cos\frac{2\pi}{9} - 1\right)\left(2\cos\frac{4\pi}{9} - 1\right)\left(2\cos\frac{8\pi}{9} - 1\right)$. [3 marks]

15. Find the value of $\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} + \cdots + \cos\frac{11\pi}{13}$. [3 marks]

16. Find the value of $8\left(\operatorname{cosec}^2\frac{\pi}{7} + \operatorname{cosec}^2\frac{2\pi}{7} + \operatorname{cosec}^2\frac{3\pi}{7}\right) - 64$. [1 mark]

17. Show that $\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$. Hence evaluate $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$. [5 marks]

18. If $\frac{x}{y} = \frac{\tan 67\frac{1}{2}^\circ}{\tan 22\frac{1}{2}^\circ}$, find $\frac{x+y}{x-y}$. [2 marks]

19. For $A = 500^\circ$, is $\cos\frac{A}{2} + \sin\frac{A}{2}$ positive or negative? [1 mark]

20. Simplify $\sqrt{1-\sin A}$ for $A = 240^\circ$ and find its value. [2 marks]

Key sheet

Q	Answer
1	$\frac{40}{9}$
2	$\sin 2A = \frac{336}{625},\ \cos 2A = -\frac{527}{625},\ \tan 2A = -\frac{336}{527}$
3	$\cos A = -\frac{\sqrt{3}}{2}$
4	$\frac{262}{343}$
5	$\sin \frac{A}{2} = \frac{\sqrt{6}}{6},\ \cos \frac{A}{2} = -\frac{\sqrt{30}}{6},\ \tan \frac{A}{2} = -\frac{\sqrt{5}}{5}$
6	$-\frac{21}{20}$
7	Proved: $\frac{1+\sin A-\cos A}{1+\sin A+\cos A} = \tan \frac{A}{2}$
8	$\frac{\sqrt{10+2\sqrt{5}}}{4}$
9	$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$; $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$
10	$2\sqrt{6} - 2\sqrt{2}$
11	$\frac{63}{64}$
12	$-\frac{2\sqrt{3}}{3}$
13	Proved: $\tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right) = 2\sec 2A$
14	1
15	$\frac{1}{2}$
16	0
17	Proved; $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$
18	$\sqrt{2}$
19	Negative
20	$-(\cos 120^\circ - \sin 120^\circ) = \frac{1+\sqrt{3}}{2}$

Solutions

1. If $\tan A = -\frac{5}{4}$, find $\tan 2A$. [1]

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$= \frac{40}{9}$

Ans: $\frac{40}{9}$

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2. If $\sin A = -\frac{24}{25}$ and $\pi < A < \frac{3\pi}{2}$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$. [3]

$\cos A = -\sqrt{1-\sin^2 A} = -\frac{7}{25}$ ($\cos A$ is negative in this quadrant)

$\sin 2A = 2\sin A \cos A = 2(-\frac{24}{25})(-\frac{7}{25}) = \frac{336}{625}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{49}{625} - \frac{576}{625} = -\frac{527}{625}$

$\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{336}{527}$

Ans: $\sin 2A = \frac{336}{625},\ \cos 2A = -\frac{527}{625},\ \tan 2A = -\frac{336}{527}$

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3. If $\cos 2A = \frac{1}{2}$ and $\pi < A < \frac{3\pi}{2}$, find $\cos A$. [2]

$\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$

$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

In this quadrant $\cos A$ is negative, so $\cos A = -\frac{\sqrt{3}}{2}$.

Ans: $\cos A = -\frac{\sqrt{3}}{2}$

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4. If $\sin A = \frac{2}{7}$, find $\sin 3A$. [2]

$\sin 3A = 3\sin A - 4\sin^3 A$

$= 3(\frac{2}{7}) - 4(\frac{2}{7})^3$

$= \frac{262}{343}$

Ans: $\frac{262}{343}$

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5. If $\cos A = \frac{2}{3}$ and $\pi < A < 2\pi$, find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$. [3]

Here $\frac{\pi}{2} < \frac{A}{2} < \pi$, so $\sin\frac{A}{2}$ is positive and $\cos\frac{A}{2}$ is negative.

$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \frac{\sqrt{6}}{6}$

$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\frac{\sqrt{30}}{6}$

$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -\frac{\sqrt{5}}{5}$

Ans: $\sin\frac{A}{2} = \frac{\sqrt{6}}{6},\ \cos\frac{A}{2} = -\frac{\sqrt{30}}{6},\ \tan\frac{A}{2} = -\frac{\sqrt{5}}{5}$

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6. If $\tan\frac{A}{2} = -\frac{3}{7}$, find $\tan A$. [1]

$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$= -\frac{21}{20}$

Ans: $-\frac{21}{20}$

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7. Prove that $\frac{1+\sin A-\cos A}{1+\sin A+\cos A}=\tan\frac{A}{2}$. [5]

$1-\cos A=2\sin^2\frac{A}{2}$, $1+\cos A=2\cos^2\frac{A}{2}$, $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$

Numerator $=2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$

Denominator $=2\cos\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)$

LHS $=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\tan\frac{A}{2}$ = RHS

Ans: Proved: $\frac{1+\sin A-\cos A}{1+\sin A+\cos A}=\tan\frac{A}{2}$

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8. Write the exact value of $\sin 108^\circ$. [1]

$\sin 108^\circ=\sin 72^\circ$

$\sin 72^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}$

Ans: $\frac{\sqrt{10+2\sqrt{5}}}{4}$

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9. Show that $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$. Hence find $\cos 36^\circ$. [5]

Let $\theta=18^\circ$. Then $5\theta=90^\circ$, so $2\theta=90^\circ-3\theta$ and $\sin 2\theta=\cos 3\theta$.

$2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$; divide by $\cos\theta\neq 0$: $2\sin\theta=4(1-\sin^2\theta)-3$

$4\sin^2\theta+2\sin\theta-1=0\Rightarrow\sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-1\pm\sqrt{5}}{4}$

18° is acute, so $\sin 18^\circ>0$ and $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$.

$\cos 36^\circ=1-2\sin^2 18^\circ=1-2\cdot\frac{6-2\sqrt{5}}{16}=\frac{\sqrt{5}+1}{4}$

Ans: $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$; $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$

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10. Find the exact value of $\tan 37\frac{1}{2}^\circ+\cot 37\frac{1}{2}^\circ$. [2]

$\tan 37\frac{1}{2}^\circ=(\sqrt{3}-\sqrt{2})(\sqrt{2}+1)$, $\cot 37\frac{1}{2}^\circ=(\sqrt{3}+\sqrt{2})(\sqrt{2}-1)$

$\tan 37\frac{1}{2}^\circ+\cot 37\frac{1}{2}^\circ=2\sqrt{6}-2\sqrt{2}$

Ans: $2\sqrt{6}-2\sqrt{2}$

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11. If $\sin 2\theta=\frac{1}{4}$, find $\sin^2\theta+\cos^4\theta$. [1]

$\sin^2\theta+\cos^4\theta=1-\frac{1}{4}\sin^2 2\theta$

$=1-\frac{1}{4}\cdot\frac{1}{16}=\frac{63}{64}$

Ans: $\frac{63}{64}$

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12. Find the value of $\cot 60^\circ-\tan 60^\circ$. [2]

$\cot A-\tan A=2\cot 2A$

With $A=60^\circ$: $\cot 60^\circ-\tan 60^\circ=-\frac{2\sqrt{3}}{3}$

Ans: $-\frac{2\sqrt{3}}{3}$

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13. Prove that $\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)=2\sec 2A$. [5]

Let $t=\tan A$: $\tan\left(\frac{\pi}{4}+A\right)=\frac{1+t}{1-t}$, $\tan\left(\frac{\pi}{4}-A\right)=\frac{1-t}{1+t}$

LHS $=\frac{(1+t)^2+(1-t)^2}{1-t^2}=\frac{2(1+t^2)}{1-t^2}$

$=\frac{2}{\cos 2A}=2\sec 2A$ = RHS

Ans: Proved: $\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)=2\sec 2A$

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14. Find the value of $\left(2\cos\frac{2\pi}{9}-1\right)\left(2\cos\frac{4\pi}{9}-1\right)\left(2\cos\frac{8\pi}{9}-1\right)$. [3]

$(2\cos\theta-1)(2\cos 2\theta-1)\cdots(2\cos 2^2\theta-1)=\frac{2\cos 2^3\theta+1}{2\cos\theta+1}$ with $\theta=\frac{2\pi}{9}$

$2^3\theta$ differs from $\pm\theta$ by a multiple of 2π, so $\cos 2^3\theta=\cos\theta$ and the value is 1.

Ans: 1

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15. Find the value of $\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}+\cdots+\cos\frac{11\pi}{13}$. [3]

$a_1=\frac{\pi}{13}$, $d=\frac{2\pi}{13}$, $n=6$, $a_n=\frac{11\pi}{13}$

$\text{Sum}=\frac{\cos\left(\frac{a_1+a_n}{2}\right)\sin\frac{nd}{2}}{\sin\frac{d}{2}}=\frac{\cos\frac{6\pi}{13}\ \sin\frac{6\pi}{13}}{\sin\frac{\pi}{13}}$

The angles in the numerator add to π, so $\cos x\sin x=\frac{1}{2}\sin 2x$ and $\sin(\pi-y)=\sin y$ cancel the denominator.

$=\frac{1}{2}$

Ans: $\frac{1}{2}$

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16. Find the value of $8\left(\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}\right)-64$. [1]

Known result: $\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}=8$

$\text{Value}=0$

Ans: 0

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17. Show that $\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta)=\frac{1}{4}\sin 3\theta$. Hence evaluate $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ$. [5]

$\sin(60^\circ+\theta)\sin(60^\circ-\theta)=\sin^2 60^\circ-\sin^2\theta=\frac{3}{4}-\sin^2\theta$

LHS $=\sin\theta\left(\frac{3}{4}-\sin^2\theta\right)=\frac{1}{4}(3\sin\theta-4\sin^3\theta)$

$=\frac{1}{4}\sin 3\theta$ = RHS

With $\theta=20^\circ$: $\sin 20^\circ\sin 40^\circ\sin 80^\circ=\frac{\sqrt{3}}{8}$

Multiply by $\sin 60^\circ=\frac{\sqrt{3}}{2}$: the value is $\frac{3}{16}$.

Ans: Proved; $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=\frac{3}{16}$

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18. If $\frac{x}{y}=\frac{\tan 67\frac{1}{2}^\circ}{\tan 22\frac{1}{2}^\circ}$, find $\frac{x+y}{x-y}$. [2]

$\frac{x+y}{x-y}=\frac{\tan A+\tan B}{\tan A-\tan B}=\frac{\sin(A+B)}{\sin(A-B)}$

$=\frac{\sin 90^\circ}{\sin 45^\circ}=\sqrt{2}$

Ans: $\sqrt{2}$

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19. For $A=500^\circ$, is $\cos\frac{A}{2}+\sin\frac{A}{2}$ positive or negative? [1]

$\frac{A}{2}=250^\circ$

$c+s>0$ for $\frac{A}{2}$ in $\left(-\frac{\pi}{4},\frac{3\pi}{4}\right)$ and <0 in $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$ (adding 2π where needed).

So $\cos 250^\circ+\sin 250^\circ$ is negative.

Ans: Negative

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20. Simplify $\sqrt{1-\sin A}$ for $A=240^\circ$ and find its value. [2]

$\sqrt{1-\sin A}=\left|\cos\frac{A}{2}-\sin\frac{A}{2}\right|$

$\frac{A}{2}=120^\circ$, where $\cos\frac{A}{2}-\sin\frac{A}{2}$ is negative.

$\sqrt{1-\sin 240^\circ}=-(\cos 120^\circ-\sin 120^\circ)=\frac{1+\sqrt{3}}{2}$

Ans: $-(\cos 120^\circ-\sin 120^\circ)=\frac{1+\sqrt{3}}{2}$
