EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 09

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. What is the rank of the matrix $\begin{bmatrix} -3 & 5 \\ -1 & -4 \end{bmatrix}$? [1 mark]

2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & -1 \\ -3 & -3 & 5 \end{bmatrix}$. [2 marks]

3. Find the rank of the matrix $\begin{bmatrix} 0 & 0 & 4 & -1 & -4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. [1 mark]

4. Reduce $A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 4 & 0 & 0 \\ 4 & 3 & 4 & -3 \end{bmatrix}$ to echelon form and find its rank. [2 marks]

5. If A is a matrix of order 2×3, what is the greatest possible value of $\operatorname{rank}(A)$? [1 mark]

6. Write the augmented matrix of the system $4x - 2z = 10$, $-x = 6$ and $3y - z = 12$. [1 mark]

7. Show that the system $2x + 5y = 10$ and $-5x + 3y = 37$ has a unique solution, and find it. [2 marks]

8. Find k so that the system $7x + y = k$, $-14x - 2y = -6$ has infinitely many solutions. [2 marks]

9. Find k so that the system $3x - 2y = 0$, $kx - 10y = 0$ has a non-trivial solution. [2 marks]

10. Does the system $4x + y + 6z = 0$, $x + y = 0$ and $x + 2z = 0$ have a non-trivial solution? [3 marks]

11. Solve the homogeneous system $x - z = 0$, $x + 3y + 2z = 0$ and $-y - z = 0$. [5 marks]

12. Show that the system $x - y + 3z = -4$, $-x - y + z = 5$ and $-3x - y - z = 4$ is inconsistent, using the rank method. [3 marks]

13. Using the rank method, show that $x - 2y + 2z = -1$, $y + z = -2$ and $4x - 2y = 12$ is consistent and find its solution. [5 marks]

14. Show that the system $x - y + 3z = 12$, $3x - 2y - 3z = -12$ and $-4x + 2y + 12z = 48$ is consistent and has infinitely many solutions. Find them. [5 marks]

15. Using Cramer's rule, solve the system $2x + 3y + 3z = -11$, $-x + 4y = -11$ and $-2x - y - z = 5$. [5 marks]

16. The point $(9, 1)$ is reflected in the line $y = x$. Find its image. [1 mark]

17. The point $(-9, 7)$ is reflected in the origin, and the image is then reflected in the y-axis. Find the final image. [2 marks]

18. The point $(4, 7)$ is reflected in the line $y = -\frac{3}{2}x$. Find the coordinates of its image. [3 marks]

19. Find the image of $(-3, 6)$ when it is rotated anticlockwise about the origin through $270°$. [1 mark]

20. The point $(-6, 9)$ is rotated anticlockwise about the origin through $150°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	2
2	2
3	2
4	3
5	2
6	$\left[\begin{array}{ccc\|c} 4 & 0 & -2 & 10 \\ -1 & 0 & 0 & 6 \\ 0 & 3 & -1 & 12 \end{array}\right]$
7	$x = -5,\ y = 4$
8	$k = 3$
9	$k = 15$
10	Yes: $\lvert A\rvert = 0$, so it has non-trivial solutions
11	$(x, y, z) = (k, -k, k)$, k any number
12	Inconsistent: $\operatorname{rank}(A) = 2 \neq \operatorname{rank}[A\lvert B] = 3$
13	$x = 3,\ y = 0,\ z = -2$
14	$(x, y, z) = (9k - 36, 12k - 48, k)$, k any number
15	$x = -1,\ y = -3,\ z = 0$
16	$(1, 9)$
17	$(-9, -7)$
18	$(-8, -1)$
19	$(6, 3)$
20	$(-\frac{9}{2} + 3\sqrt{3}, -\frac{9\sqrt{3}}{2} - 3)$

Solutions

1. What is the rank of the matrix $\begin{bmatrix} -3 & 5 \\ -1 & -4 \end{bmatrix}$? [1]

$\det A = (-3)(-4) - (5)(-1) = 17$

$\det A \neq 0$, so A itself is a non-singular submatrix of order 2.

$\text{rank}(A) = 2$

Ans: 2

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2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & -1 \\ -3 & -3 & 5 \end{bmatrix}$. [2]

$|A| = (-5-3) - (-5-3) - 3(3-3) = 0$

$|A| = 0$, so the rank is less than 3.

The submatrix $\begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix}$ has determinant $-4 \neq 0$.

$\therefore \text{rank}(A) = 2$

Ans: 2

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3. Find the rank of the matrix $\begin{bmatrix} 0 & 0 & 4 & -1 & -4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. [1]

The non-zero rows begin with 2, 4 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 2$

Ans: 2

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4. Reduce $A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 4 & 0 & 0 \\ 4 & 3 & 4 & -3 \end{bmatrix}$ to echelon form and find its rank. [2]

Apply $R_3 \to R_3 - 4R_1$: $A \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 11 & 0 & -15 \end{bmatrix}$

Apply $R_3 \to 4R_3 - 11R_2$: $\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -60 \end{bmatrix}$

This is in echelon form with 3 non-zero rows.

$\therefore \text{rank}(A) = 3$

Ans: 3

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5. If A is a matrix of order 2×3, what is the greatest possible value of $\text{rank}(A)$? [1]

For an $m \times n$ matrix, $\text{rank} \leq \min(m, n)$.

$\min(2, 3) = 2$

Ans: 2

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6. Write the augmented matrix of the system $4x - 2z = 10$, $-x = 6$ and $3y - z = 12$. [1]

Coefficient matrix $A = \begin{bmatrix} 4 & 0 & -2 \\ -1 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} 10 \\ 6 \\ 12 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} 4 & 0 & -2 & 10 \\ -1 & 0 & 0 & 6 \\ 0 & 3 & -1 & 12 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} 4 & 0 & -2 & 10 \\ -1 & 0 & 0 & 6 \\ 0 & 3 & -1 & 12 \end{array}\right]$

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7. Show that the system $2x + 5y = 10$ and $-5x + 3y = 37$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{2}{-5} \neq \frac{b_1}{b_2} = \frac{5}{3}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{31}\begin{bmatrix} 3 & -5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 37 \end{bmatrix}$

$x = -5,\ y = 4$

Ans: $x = -5,\ y = 4$

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8. Find k so that the system $7x + y = k$, $-14x - 2y = -6$ has infinitely many solutions. [2]

Infinitely many solutions need $\frac{7}{-14} = \frac{1}{-2} = \frac{k}{-6}$

$\frac{k}{-6} = \frac{1}{-2} \Rightarrow k = 3$

Ans: $k = 3$

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9. Find k so that the system $3x - 2y = 0$, $kx - 10y = 0$ has a non-trivial solution. [2]

A non-trivial solution needs $\frac{3}{k} = \frac{-2}{-10}$

$k = 15$

Ans: $k = 15$

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10. Does the system $4x + y + 6z = 0$, $x + y = 0$ and $x + 2z = 0$ have a non-trivial solution? [3]

$|A| = 4(2 - 0) - (2 - 0) + 6(0 - 1) = 0$

$|A| = 0$, so the system has infinitely many solutions, including non-trivial ones.

Ans: Yes: $|A| = 0$, so it has non-trivial solutions

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11. Solve the homogeneous system $x - z = 0$, $x + 3y + 2z = 0$ and $-y - z = 0$. [5]

$|A| = (-3 + 2) + 0(-1 - 0) - (-1 - 0) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = k$ in the first two equations:

$x = k$, $x + 3y = -2k$

Solving (determinant $3 \neq 0$): $x = k,\ y = -k$

These values also satisfy the third equation.

$(x, y, z) = (k, -k, k)$, k any number

Ans: $(x, y, z) = (k, -k, k)$, k any number

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12. Show that the system $x - y + 3z = -4$, $-x - y + z = 5$ and $-3x - y - z = 4$ is inconsistent, using the rank method. [3]

Apply $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 3R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & -2 & 4 & 1 \\ 0 & -4 & 8 & -8 \end{array}\right]$

Apply $R_3 \to R_3 - 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & -10 \end{array}\right]$

$\text{rank}(A) = 2$ (non-zero rows of the coefficient part), $\text{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A|B] = 3$

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13. Using the rank method, show that $x - 2y + 2z = -1$, $y + z = -2$ and $4x - 2y = 12$ is consistent and find its solution. [5]

Apply $R_3 \to R_3 - 4R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 6 & -8 & 16 \end{array}\right]$

Apply $R_3 \to R_3 - 6R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -14 & 28 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $-14z = 28$ gives $z = -2$

From row 2: $y + z = -2$, so $y = 0$

From row 1: $x - 2y + 2z = -1$, so $x = 3$

$x = 3,\ y = 0,\ z = -2$

Ans: $x = 3,\ y = 0,\ z = -2$

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14. Show that the system $x - y + 3z = 12$, $3x - 2y - 3z = -12$ and $-4x + 2y + 12z = 48$ is consistent and has infinitely many solutions. Find them. [5]

Apply $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 4R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 12 \\ 0 & 1 & -12 & -48 \\ 0 & -2 & 24 & 96 \end{array}\right]$

Apply $R_3 \to R_3 + 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 12 \\ 0 & 1 & -12 & -48 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $y - 12z = -48$, so $y = 12k - 48$

From row 1: $x - y + 3z = 12$, so $x = 9k - 36$

$(x, y, z) = (9k - 36, 12k - 48, k)$, k any number

Ans: $(x, y, z) = (9k - 36, 12k - 48, k)$, k **any number**

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15. Using Cramer's rule, solve the system $2x + 3y + 3z = -11$, $-x + 4y = -11$ and $-2x - y - z = 5$. [5]

$$\Delta = \begin{vmatrix} 2 & 3 & 3 \\ -1 & 4 & 0 \\ -2 & -1 & -1 \end{vmatrix} = 2(-4-0) - 3(1-0) + 3(1+8) = 16$$

$$\Delta_1 = \begin{vmatrix} -11 & 3 & 3 \\ -11 & 4 & 0 \\ 5 & -1 & -1 \end{vmatrix} = -11(-4-0) - 3(11-0) + 3(11-20) = -16$$

$$\Delta_2 = \begin{vmatrix} 2 & -11 & 3 \\ -1 & -11 & 0 \\ -2 & 5 & -1 \end{vmatrix} = 2(11-0) + 11(1-0) + 3(-5-22) = -48$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & -11 \\ -1 & 4 & -11 \\ -2 & -1 & 5 \end{vmatrix} = 2(20-11) - 3(-5-22) - 11(1+8) = 0$$

$x = \frac{\Delta_1}{\Delta} = \frac{-16}{16} = -1,\ y = \frac{\Delta_2}{\Delta} = \frac{-48}{16} = -3,\ z = \frac{\Delta_3}{\Delta} = \frac{0}{16} = 0$

Ans: $x = -1,\ y = -3,\ z = 0$

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16. The point $(9, 1)$ is reflected in the line $y = x$. Find its image. [1]

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

Ans: $(1, 9)$

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17. The point $(-9, 7)$ is reflected in the origin, and the image is then reflected in the y-axis. Find the final image. [2]

Reflection in the origin: $(-9, 7) \mapsto (9, -7)$

Reflection in the y-axis: $(9, -7) \mapsto (-9, -7)$

In matrix form: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -9 \\ 7 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}$

Ans: $(-9, -7)$

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18. The point $(4, 7)$ is reflected in the line $y = -\frac{3}{2}x$. Find the coordinates of its image. [3]

$\tan\theta = m = -\frac{3}{2}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = -\frac{5}{13},\ \sin 2\theta = \frac{2m}{1+m^2} = -\frac{12}{13}$

Reflection matrix: $\begin{bmatrix} -\frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix}$

$$\begin{bmatrix} -\frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \end{bmatrix}$$

Image: $(-8, -1)$

Ans: $(-8, -1)$

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19. Find the image of $(-3, 6)$ when it is rotated anticlockwise about the origin through $270°$. [1]

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Ans: $(6, 3)$

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20. The point $(-6, 9)$ is rotated anticlockwise about the origin through $150°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -6 \\ 9 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} + 3\sqrt{3} \\ -\frac{9\sqrt{3}}{2} - 3 \end{bmatrix}$$

Image: $(-\frac{9}{2} + 3\sqrt{3}, -\frac{9\sqrt{3}}{2} - 3)$

Ans: $(-\frac{9}{2} + 3\sqrt{3}, -\frac{9\sqrt{3}}{2} - 3)$