EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 08

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. [1 mark]

2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 3 & 4 & 2 \\ -1 & 2 & 1 \\ 1 & 3 & -3 \end{bmatrix}$. [2 marks]

3. Find the rank of the matrix $\begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$. [1 mark]

4. Reduce $A = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & -2 & -3 & -3 \\ -2 & -6 & 3 & -3 \end{bmatrix}$ to echelon form and find its rank. [2 marks]

5. If $\text{rank}(A) = 2$, $\text{rank}(B) = 5$ and AB is defined, what is the greatest possible value of $\text{rank}(AB)$? [1 mark]

6. Write the augmented matrix of the system $3x + y + 2z = -9$, $-x + z = -9$ and $-2y + 6z = 8$. [1 mark]

7. Show that the system $-5x + 3y = 10$ and $3x + 4y = -6$ has a unique solution, and find it. [2 marks]

8. Show that the system $-6x + 5y = -10$ and $5x - 2y = 17$ has a unique solution, and find it. [2 marks]

9. Find k so that the system $8x + 7y = 0$, $kx - 14y = 0$ has a non-trivial solution. [2 marks]

10. Examine whether the homogeneous system $4x - 3y - 3z = 0$, $5x - 3y - 4z = 0$ and $-2x - 9y + 5z = 0$ has only the trivial solution. [3 marks]

11. Show that the system $-x - 3y = 0$, $-5x - 3y - 6z = 0$ and $3x + 7y + z = 0$ has a non-trivial solution, and find all its solutions. [5 marks]

12. Show that the system $x + 3y - 2z = -1$, $-2x - 2y = 3$ and $4x + 8y - 4z = 1$ is inconsistent, using the rank method. [3 marks]

13. Examine the consistency of the system $x + 2z = 3$, $2x + 3y - z = -15$ and $-3x - y - 3z = 2$ by the rank method and solve it if consistent. [5 marks]

14. Solve $x - 2y - z = 1$, $y - 2z = -2$ and $x - 3y + z = 3$ by the rank method. [5 marks]

15. Using Cramer's rule, solve the system $-x - 3z = 3$, $-x + z = -5$ and $4x + 4y - z = 2$. [5 marks]

16. The point $(-7, 5)$ is reflected in the line $y = x$. Find its image. [1 mark]

17. Under reflection in the origin, the image of a point P is $(9, -8)$. Find P. [2 marks]

18. Find the image of $(6, 10)$ under reflection in the line $y = \frac{5}{12}x$. [3 marks]

19. Find the image of $(2, 9)$ when it is rotated anticlockwise about the origin through $90°$. [1 mark]

20. The axes are rotated through $120°$ about the origin, the point staying fixed. Find the new coordinates of the point $(-7, -6)$. [3 marks]

Key sheet

Q	Answer
1	2
2	3
3	2
4	2
5	2
6	$\left[\begin{array}{ccc\|c} 3 & 1 & 2 & -9 \\ -1 & 0 & 1 & -9 \\ 0 & -2 & 6 & 8 \end{array}\right]$
7	$x = -2,\ y = 0$
8	$x = 5,\ y = 4$
9	$k = -16$
10	Yes: $\lvert A\rvert = 0$, so it has non-trivial solutions
11	$(x, y, z) = (-3k, k, 2k)$, k any number
12	Inconsistent: $\operatorname{rank}(A) = 2 \neq \operatorname{rank}[A\lvert B] = 3$
13	$x = -3,\ y = -2,\ z = 3$
14	$(x, y, z) = (5k - 3, 2k - 2, k)$, k any number
15	$x = 3,\ y = -3,\ z = -2$
16	$(5, -7)$
17	$P(-9, 8)$
18	$(\frac{1914}{169}, -\frac{470}{169})$
19	$(-9, 2)$
20	$(\frac{7}{2} - 3\sqrt{3}, 3 + \frac{7\sqrt{3}}{2})$

Solutions

1. Find the rank of $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. [1]

$\det A = (2)(3) - (-1)(1) = 7$

$\det A \neq 0$, so A itself is a non-singular submatrix of order 2.

$\text{rank}(A) = 2$

Ans: 2

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2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 3 & 4 & 2 \\ -1 & 2 & 1 \\ 1 & 3 & -3 \end{bmatrix}$. [2]

$|A| = 3(-6-3) - 4(3-1) + 2(-3-2) = -45$

$|A| \neq 0$, so A is non-singular.

$\text{rank}(A) = 3$

Ans: 3

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3. Find the rank of the matrix $\begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$. [1]

The non-zero rows begin with 0, 2 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 2$

Ans: 2

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4. Reduce $A = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & -2 & -3 & -3 \\ -2 & -6 & 3 & -3 \end{bmatrix}$ to echelon form and find its rank. [2]

Apply $R_3 \to R_3 + 2R_1$: $A \sim \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & -2 & -3 & -3 \\ 0 & 2 & 3 & 3 \end{bmatrix}$

Apply $R_3 \to R_3 + R_2$: $\sim \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & -2 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in echelon form with 2 non-zero rows.

$\therefore \text{rank}(A) = 2$

Ans: 2

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5. If $\text{rank}(A) = 2$, $\text{rank}(B) = 5$ and AB is defined, what is the greatest possible value of $\text{rank}(AB)$? [1]

$\text{rank}(AB)$ cannot exceed the rank of either A or B.

$\text{rank}(AB) \leq \min(2, 5) = 2$

Ans: 2

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6. Write the augmented matrix of the system $3x + y + 2z = -9$, $-x + z = -9$ and $-2y + 6z = 8$. [1]

Coefficient matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & -2 & 6 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} -9 \\ -9 \\ 8 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} 3 & 1 & 2 & -9 \\ -1 & 0 & 1 & -9 \\ 0 & -2 & 6 & 8 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} 3 & 1 & 2 & -9 \\ -1 & 0 & 1 & -9 \\ 0 & -2 & 6 & 8 \end{array}\right]$

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7. Show that the system $-5x + 3y = 10$ and $3x + 4y = -6$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{-5}{3} \neq \frac{b_1}{b_2} = \frac{3}{4}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{-29}\begin{bmatrix} 4 & -3 \\ -3 & -5 \end{bmatrix}\begin{bmatrix} 10 \\ -6 \end{bmatrix}$

$x = -2,\ y = 0$

Ans: $x = -2,\ y = 0$

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8. Show that the system $-6x + 5y = -10$ and $5x - 2y = 17$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{-6}{5} \neq \frac{b_1}{b_2} = \frac{5}{-2}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{-13}\begin{bmatrix} -2 & -5 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} -10 \\ 17 \end{bmatrix}$

$x = 5,\ y = 4$

Ans: $x = 5,\ y = 4$

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9. Find k so that the system $8x + 7y = 0$, $kx - 14y = 0$ has a non-trivial solution. [2]

A non-trivial solution needs $\frac{8}{k} = \frac{7}{-14}$

$k = -16$

Ans: $k = -16$

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10. Examine whether the homogeneous system $4x - 3y - 3z = 0$, $5x - 3y - 4z = 0$ and $-2x - 9y + 5z = 0$ has only the trivial solution. [3]

$|A| = 4(-15 - 36) + 3(25 - 8) - 3(-45 - 6) = 0$

$|A| = 0$, so the system has infinitely many solutions, including non-trivial ones.

Ans: Yes: $|A| = 0$, so it has non-trivial solutions

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11. Show that the system $-x-3y=0$, $-5x-3y-6z=0$ and $3x+7y+z=0$ has a non-trivial solution, and find all its solutions. [5]

$|A| = -(-3+42)+3(-5+18)+0(-35+9)=0$

$|A|=0$, so there are non-trivial solutions.

Put $z=2k$ in the first two equations:

$-x-3y=0$, $-5x-3y=12k$

Solving (determinant $-12\neq 0$): $x=-3k,\ y=k$

These values also satisfy the third equation.

$(x,y,z)=(-3k,k,2k)$, k any number

Ans: $(x,y,z)=(-3k,k,2k)$**, k any number**

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12. Show that the system $x+3y-2z=-1$, $-2x-2y=3$ and $4x+8y-4z=1$ is inconsistent, using the rank method. [3]

Apply $R_2\to R_2+2R_1$, $R_3\to R_3-4R_1$: $[A\,|\,B]\sim\left[\begin{array}{ccc|c}1&3&-2&-1\\0&4&-4&1\\0&-4&4&5\end{array}\right]$

Apply $R_3\to R_3+R_2$: $\sim\left[\begin{array}{ccc|c}1&3&-2&-1\\0&4&-4&1\\0&0&0&6\end{array}\right]$

$\mathrm{rank}(A)=2$ (non-zero rows of the coefficient part), $\mathrm{rank}[A\,|\,B]=3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\mathrm{rank}(A)=2\neq\mathrm{rank}[A|B]=3$

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13. Examine the consistency of the system $x+2z=3$, $2x+3y-z=-15$ and $-3x-y-3z=2$ by the rank method and solve it if consistent. [5]

Apply $R_2\to R_2-2R_1$, $R_3\to R_3+3R_1$: $[A\,|\,B]\sim\left[\begin{array}{ccc|c}1&0&2&3\\0&3&-5&-21\\0&-1&3&11\end{array}\right]$

Apply $R_2\leftrightarrow R_3$: $\sim\left[\begin{array}{ccc|c}1&0&2&3\\0&-1&3&11\\0&3&-5&-21\end{array}\right]$

Apply $R_3\to R_3+3R_2$: $\sim\left[\begin{array}{ccc|c}1&0&2&3\\0&-1&3&11\\0&0&4&12\end{array}\right]$

$\mathrm{rank}(A)=\mathrm{rank}[A\,|\,B]=3$ = number of unknowns: consistent, unique solution.

From row 3: $4z=12$ gives $z=3$

From row 2: $-y+3z=11$, so $y=-2$

From row 1: $x+2z=3$, so $x=-3$

$x=-3,\ y=-2,\ z=3$

Ans: $x=-3,\ y=-2,\ z=3$

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14. Solve $x - 2y - z = 1$, $y - 2z = -2$ and $x - 3y + z = 3$ by the rank method. [5]

Apply $R_3 \to R_3 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & 2 & 2 \end{array}\right]$

Apply $R_3 \to R_3 + R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $y - 2z = -2$, so $y = 2k - 2$

From row 1: $x - 2y - z = 1$, so $x = 5k - 3$

$(x, y, z) = (5k - 3, 2k - 2, k)$, k any number

Ans: $(x, y, z) = (5k - 3, 2k - 2, k)$, k **any number**

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15. Using Cramer's rule, solve the system $-x - 3z = 3$, $-x + z = -5$ and $4x + 4y - z = 2$. [5]

$$\Delta = \begin{vmatrix} -1 & 0 & -3 \\ -1 & 0 & 1 \\ 4 & 4 & -1 \end{vmatrix} = -(0 - 4) + 0(1 - 4) - 3(-4 - 0) = 16$$

$$\Delta_1 = \begin{vmatrix} 3 & 0 & -3 \\ -5 & 0 & 1 \\ 2 & 4 & -1 \end{vmatrix} = 3(0 - 4) + 0(5 - 2) - 3(-20 - 0) = 48$$

$$\Delta_2 = \begin{vmatrix} -1 & 3 & -3 \\ -1 & -5 & 1 \\ 4 & 2 & -1 \end{vmatrix} = -(5 - 2) - 3(1 - 4) - 3(-2 + 20) = -48$$

$$\Delta_3 = \begin{vmatrix} -1 & 0 & 3 \\ -1 & 0 & -5 \\ 4 & 4 & 2 \end{vmatrix} = -(0 + 20) + 0(-2 + 20) + 3(-4 - 0) = -32$$

$x = \frac{\Delta_1}{\Delta} = \frac{48}{16} = 3,\ y = \frac{\Delta_2}{\Delta} = \frac{-48}{16} = -3,\ z = \frac{\Delta_3}{\Delta} = \frac{-32}{16} = -2$

Ans: $x = 3,\ y = -3,\ z = -2$

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16. The point $(-7, 5)$ is reflected in the line $y = x$. Find its image. [1]

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

Ans: $(5, -7)$

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17. Under reflection in the origin, the image of a point P is $(9, -8)$. Find P. [2]

Reflecting twice brings a point back, so reflect the image again.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ -8 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

$P = (-9, 8)$

Ans: $P(-9, 8)$

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18. Find the image of $(6, 10)$ under reflection in the line $y = \frac{5}{12}x$. [3]

$\tan\theta = m = \frac{5}{12}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = \frac{119}{169}$, $\sin 2\theta = \frac{2m}{1+m^2} = \frac{120}{169}$

Reflection matrix: $\begin{bmatrix} \frac{119}{169} & \frac{120}{169} \\ \frac{120}{169} & -\frac{119}{169} \end{bmatrix}$

$$\begin{bmatrix} \frac{119}{169} & \frac{120}{169} \\ \frac{120}{169} & -\frac{119}{169} \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{1914}{169} \\ -\frac{470}{169} \end{bmatrix}$$

Image: $(\frac{1914}{169}, -\frac{470}{169})$

Ans: $(\frac{1914}{169}, -\frac{470}{169})$

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19. Find the image of $(2, 9)$ when it is rotated anticlockwise about the origin through $90°$. [1]

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$$

Ans: $(-9, 2)$

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20. The axes are rotated through $120°$ about the origin, the point staying fixed. Find the new coordinates of the point $(-7, -6)$. [3]

Turning the axes uses the transpose: $x' = x\cos\theta + y\sin\theta,\ y' = -x\sin\theta + y\cos\theta$

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -7 \\ -6 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} - 3\sqrt{3} \\ 3 + \frac{7\sqrt{3}}{2} \end{bmatrix}$$

New coordinates: $(\frac{7}{2} - 3\sqrt{3}, 3 + \frac{7\sqrt{3}}{2})$

Ans: $(\frac{7}{2} - 3\sqrt{3}, 3 + \frac{7\sqrt{3}}{2})$
