EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 2 & 2 \\ -2 & 5 \end{bmatrix}$. [1 mark]

2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 2 \\ -2 & -3 & 4 \end{bmatrix}$. [2 marks]

3. The matrix $\begin{bmatrix} 6 & -1 & 6 & 5 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ is in echelon form. Write its rank. [1 mark]

4. Find the rank of $A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{bmatrix}$ using elementary row transformations. [2 marks]

5. What is the rank of the identity matrix I_3? [1 mark]

6. Write down the system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} -5 & -3 & 1 & 5 \\ 4 & 6 & 1 & 5 \\ -4 & 2 & 0 & 3 \end{array}\right]$. [1 mark]

7. Show that the system $2x - 5y = -19$ and $3x - 2y = -1$ has a unique solution, and find it. [2 marks]

8. Show that the system $7x - 5y = 17$ and $-2x + y = -7$ has a unique solution, and find it. [2 marks]

9. Show that the system $x + 5y = 0$, $-3x - 4y = 0$ has only the trivial solution. [2 marks]

10. Does the system $-2x = 0$, $4x + y + 2z = 0$ and $-2x + y + z = 0$ have a non-trivial solution? [3 marks]

11. Show that the system $-3x - 2y - 3z = 0$, $2x + y + 3z = 0$ and $-y + 3z = 0$ has a non-trivial solution, and find all its solutions. [5 marks]

12. Examine the consistency of $x - z = -6$, $-4x + 2y + 4z = 6$ and $3x - 2y - 3z = 3$ by the rank method. [3 marks]

13. Examine the consistency of the system $x + 4y = 13$, $x + y - 3z = 7$ and $-3x - 3y - z = -11$ by the rank method and solve it if consistent. [5 marks]

14. Solve $x - y - 2z = 4$, $y + z = 0$ and $-x + 2y + 3z = -4$ by the rank method. [5 marks]

15. Using Cramer's rule, solve the system $2y + z = 3$, $2x - 3y - z = 3$ and $2y + 4z = 12$. [5 marks]

16. The point $(9, 5)$ is reflected in the y-axis. Find its image. [1 mark]

17. The point $(6, 4)$ is reflected in the line $y = -x$, and the image is then reflected in the y-axis. Find the final image. [2 marks]

18. The point $(-6, -7)$ is reflected in the line $y = \sqrt{3}x$. Find the coordinates of its image. [3 marks]

19. Find the image of $(-4, -2)$ when it is rotated anticlockwise about the origin through $90°$. [1 mark]

20. The point $(-6, 8)$ is rotated anticlockwise about the origin through $120°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	2
2	3
3	4
4	1
5	3
6	$-5x-3y+z=5$, $4x+6y+z=5$ and $-4x+2y=3$
7	$x=3,\ y=5$
8	$x=6,\ y=5$
9	Only $x=0, y=0$
10	No: only the trivial solution ($\lvert A\rvert=2\neq 0$)
11	$(x,y,z)=(3k,-3k,-k)$, k any number
12	Inconsistent: $\text{rank}(A)=2\neq\text{rank}[A\lvert B]=3$
13	$x=1,\ y=3,\ z=-1$
14	$(x,y,z)=(k+4,-k,k)$, k any number
15	$x=3,\ y=0,\ z=3$
16	$(-9,5)$
17	$(4,-6)$
18	$(3-\frac{7\sqrt{3}}{2},-3\sqrt{3}-\frac{7}{2})$
19	$(2,-4)$
20	$(3-4\sqrt{3},-3\sqrt{3}-4)$

Solutions

1. Find the rank of $A = \begin{bmatrix} 2 & 2 \\ -2 & 5 \end{bmatrix}$. [1]

$\det A = (2)(5) - (2)(-2) = 14$

$\det A \neq 0$, so A itself is a non-singular submatrix of order 2.

$\text{rank}(A) = 2$

Ans: 2

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2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 2 \\ -2 & -3 & 4 \end{bmatrix}$. [2]

$|A| = 3(-8+6) + (0+4) + 4(0-4) = -18$

$|A| \neq 0$, so A is non-singular.

$\text{rank}(A) = 3$

Ans: 3

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3. The matrix $\begin{bmatrix} 6 & -1 & 6 & 5 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ is in echelon form. Write its rank. [1]

The non-zero rows begin with 0, 1, 2, 3 zeros: the count increases row by row.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 4$

Ans: 4

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4. Find the rank of $A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{bmatrix}$ using elementary row transformations. [2]

Apply $R_2 \to R_2 + 2R_1$: $A \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

This is in echelon form with 1 non-zero row.

$\therefore \text{rank}(A) = 1$

Ans: 1

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5. What is the rank of the identity matrix I_3? [1]

I_3 is in echelon form with 3 non-zero rows.

$\text{rank}(I_3) = 3$

Ans: 3

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6. Write down the system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} -5 & -3 & 1 & 5 \\ 4 & 6 & 1 & 5 \\ -4 & 2 & 0 & 3 \end{array}\right]$. [1]

Each row gives one equation: the first three entries are the coefficients of x, y, z, the last is the constant.

$-5x - 3y + z = 5$, $4x + 6y + z = 5$ and $-4x + 2y = 3$

Ans: $-5x - 3y + z = 5$, $4x + 6y + z = 5$ **and** $-4x + 2y = 3$

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7. Show that the system $2x - 5y = -19$ and $3x - 2y = -1$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{2}{3} \neq \frac{b_1}{b_2} = \frac{-5}{-2}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{11}\begin{bmatrix} -2 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -19 \\ -1 \end{bmatrix}$

$x = 3,\ y = 5$

Ans: $x = 3,\ y = 5$

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8. Show that the system $7x - 5y = 17$ and $-2x + y = -7$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{7}{-2} \neq \frac{b_1}{b_2} = \frac{-5}{1}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{-3}\begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 17 \\ -7 \end{bmatrix}$

$x = 6,\ y = 5$

Ans: $x = 6,\ y = 5$

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9. Show that the system $x + 5y = 0$, $-3x - 4y = 0$ has only the trivial solution. [2]

$\frac{a_1}{a_2} = \frac{1}{-3},\ \frac{b_1}{b_2} = \frac{5}{-4}$

These are not equal (equivalently $|A| = 11 \neq 0$).

So the only solution is $x = y = 0$.

Ans: Only $x = 0, y = 0$

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10. Does the system $-2x = 0$, $4x + y + 2z = 0$ and $-2x + y + z = 0$ have a non-trivial solution? [3]

$|A| = -2(1 - 2) + 0(4 + 4) + 0(4 + 2) = 2$

$|A| \neq 0$, so the system has only the trivial solution $x = y = z = 0$.

Ans: No: only the trivial solution ($|A| = 2 \neq 0$)

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11. Show that the system $-3x - 2y - 3z = 0$, $2x + y + 3z = 0$ and $-y + 3z = 0$ has a non-trivial solution, and find all its solutions. [5]

$|A| = -3(3 + 3) + 2(6 - 0) - 3(-2 - 0) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = -k$ in the first two equations:

$-3x - 2y = -3k$, $2x + y = 3k$

Solving (determinant $1 \neq 0$): $x = 3k,\ y = -3k$

These values also satisfy the third equation.

$(x, y, z) = (3k, -3k, -k)$, k any number

Ans: $(x, y, z) = (3k, -3k, -k)$, k any number

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12. Examine the consistency of $x - z = -6$, $-4x + 2y + 4z = 6$ and $3x - 2y - 3z = 3$ by the rank method. [3]

Apply $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 - 3R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 2 & 0 & -18 \\ 0 & -2 & 0 & 21 \end{array}\right]$

Apply $R_3 \to R_3 + R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 2 & 0 & -18 \\ 0 & 0 & 0 & 3 \end{array}\right]$

$\text{rank}(A) = 2$ (non-zero rows of the coefficient part), $\text{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A|B] = 3$

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13. Examine the consistency of the system $x + 4y = 13$, $x + y - 3z = 7$ and $-3x - 3y - z = -11$ by the rank method and solve it if consistent. [5]

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 3R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 4 & 0 & 13 \\ 0 & -3 & -3 & -6 \\ 0 & 9 & -1 & 28 \end{array}\right]$

Apply $R_3 \to R_3 + 3R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 4 & 0 & 13 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & -10 & 10 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $-10z = 10$ gives $z = -1$

From row 2: $-3y - 3z = -6$, so $y = 3$

From row 1: $x + 4y = 13$, so $x = 1$

$x = 1,\ y = 3,\ z = -1$

Ans: $x = 1,\ y = 3,\ z = -1$

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14. Solve $x - y - 2z = 4$, $y + z = 0$ and $-x + 2y + 3z = -4$ by the rank method. [5]

Apply $R_3 \to R_3 + R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

Apply $R_3 \to R_3 - R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $y + z = 0$, so $y = -k$

From row 1: $x - y - 2z = 4$, so $x = k + 4$

$(x, y, z) = (k + 4, -k, k)$, k any number

Ans: $(x, y, z) = (k + 4, -k, k)$, k **any number**

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15. Using Cramer's rule, solve the system $2y + z = 3$, $2x - 3y - z = 3$ and $2y + 4z = 12$. [5]

$$\Delta = \begin{vmatrix} 0 & 2 & 1 \\ 2 & -3 & -1 \\ 0 & 2 & 4 \end{vmatrix} = 0(-12+2) - 2(8-0) + (4-0) = -12$$

$$\Delta_1 = \begin{vmatrix} 3 & 2 & 1 \\ 3 & -3 & -1 \\ 12 & 2 & 4 \end{vmatrix} = 3(-12+2) - 2(12+12) + (6+36) = -36$$

$$\Delta_2 = \begin{vmatrix} 0 & 3 & 1 \\ 2 & 3 & -1 \\ 0 & 12 & 4 \end{vmatrix} = 0(12+12) - 3(8-0) + (24-0) = 0$$

$$\Delta_3 = \begin{vmatrix} 0 & 2 & 3 \\ 2 & -3 & 3 \\ 0 & 2 & 12 \end{vmatrix} = 0(-36-6) - 2(24-0) + 3(4-0) = -36$$

$x = \frac{\Delta_1}{\Delta} = \frac{-36}{-12} = 3,\ y = \frac{\Delta_2}{\Delta} = \frac{0}{-12} = 0,\ z = \frac{\Delta_3}{\Delta} = \frac{-36}{-12} = 3$

Ans: $x = 3,\ y = 0,\ z = 3$

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16. The point $(9, 5)$ is reflected in the y-axis. Find its image. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$$

Ans: $(-9, 5)$

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17. The point $(6, 4)$ is reflected in the line $y = -x$, and the image is then reflected in the y-axis. Find the final image. [2]

Reflection in the line $y = -x$: $(6, 4) \mapsto (-4, -6)$

Reflection in the y-axis: $(-4, -6) \mapsto (4, -6)$

In matrix form: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

Ans: $(4, -6)$

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18. The point $(-6, -7)$ is reflected in the line $y = \sqrt{3}x$. Find the coordinates of its image. [3]

$\tan\theta = m = \sqrt{3}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = -\frac{1}{2},\ \sin 2\theta = \frac{2m}{1+m^2} = \frac{\sqrt{3}}{2}$

Reflection matrix: $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -6 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 - \frac{7\sqrt{3}}{2} \\ -3\sqrt{3} - \frac{7}{2} \end{bmatrix}$$

Image: $(3 - \frac{7\sqrt{3}}{2}, -3\sqrt{3} - \frac{7}{2})$

Ans: $(3 - \frac{7\sqrt{3}}{2}, -3\sqrt{3} - \frac{7}{2})$

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19. Find the image of $(-4, -2)$ when it is rotated anticlockwise about the origin through $90°$. [1]

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Ans: $(2, -4)$

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20. The point $(-6, 8)$ is rotated anticlockwise about the origin through $120°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 - 4\sqrt{3} \\ -3\sqrt{3} - 4 \end{bmatrix}$

Image: $(3 - 4\sqrt{3}, -3\sqrt{3} - 4)$

Ans: $(3 - 4\sqrt{3}, -3\sqrt{3} - 4)$