EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 06

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 5 & -2 \\ 6 & -5 \end{bmatrix}$. [1 mark]

2. Find the rank of $A = \begin{bmatrix} 1 & -1 & -1 \\ -3 & 3 & -3 \\ 2 & -2 & 4 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2 marks]

3. Find the rank of the matrix $\begin{bmatrix} 0 & 4 & 5 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. [1 mark]

4. Reduce $A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ -2 & 1 & 0 & 4 \\ 1 & 2 & -1 & 3 \end{bmatrix}$ to echelon form and find its rank. [2 marks]

5. A and B are square matrices of order 7 with $\text{rank}(A) = 1$ and $\text{rank}(B) = 2$. What is the greatest possible value of $\text{rank}(A + B)$? [1 mark]

6. Write down the system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} 6 & 3 & -5 & 0 \\ -2 & 0 & -5 & -5 \\ 5 & 0 & 3 & 12 \end{array}\right]$. [1 mark]

7. Show that the system $-6x + 2y = 8$ and $7x - 4y = -11$ has a unique solution, and find it. [2 marks]

8. How many solutions does the system $5x - y = 6$ and $-15x + 3y = -13$ have? [2 marks]

9. Show that the system $2x - 3y = 0$, $3x + 6y = 0$ has only the trivial solution. [2 marks]

10. Examine whether the homogeneous system $x - y + 6z = 0$, $2x + 5y - 9z = 0$ and $-5x - 4y - 3z = 0$ has only the trivial solution. [3 marks]

11. Show that the system $2x + 3y + 2z = 0$, $x - 3y - 2z = 0$ and $-3x + 6y + 4z = 0$ has a non-trivial solution, and find all its solutions. [5 marks]

12. Examine the consistency of $x + 4z = -3$, $x + 2y - 2z = -6$ and $-y + 3z = -2$ by the rank method. [3 marks]

13. Examine the consistency of the system $x - z = 2$, $4x + 2y - z = 9$ and $-3z = 3$ by the rank method and solve it if consistent. [5 marks]

14. Solve $x - y - 3z = -12$, $-y - z = -6$ and $x - 2y - 4z = -18$ by the rank method. [5 marks]

15. Solve $x + 2y + z = 2$, $-2x - 3y - z = -4$ and $-3y = -3$ by Cramer's rule. [5 marks]

16. The point $(-2, 7)$ is reflected in the origin. Find its image. [1 mark]

17. Under reflection in the x-axis, the image of a point P is $(-1, 4)$. Find P. [2 marks]

18. The point $(8, -3)$ is reflected in the line $y = \sqrt{3}x$. Find the coordinates of its image. [3 marks]

19. Find the image of $(7, -4)$ when it is rotated anticlockwise about the origin through $180°$. [1 mark]

20. The point $(10, -3)$ is rotated anticlockwise about the origin through $30°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	2
2	2
3	3
4	3
5	3
6	$6x + 3y - 5z = 0$, $-2x - 5z = -5$ and $5x + 3z = 12$
7	$x = -1,\ y = 1$
8	Inconsistent: no solution
9	Only $x = 0, y = 0$
10	Yes: $\lvert A\rvert = 0$, so it has non-trivial solutions
11	$(x, y, z) = (0, -2k, 3k)$, k any number
12	Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A\lvert B] = 3$
13	$x = 1,\ y = 2,\ z = -1$
14	$(x, y, z) = (2k - 6, 6 - k, k)$, k any number
15	$x = 1,\ y = 1,\ z = -1$
16	$(2, -7)$
17	$P(-1, -4)$
18	$(-4 - \frac{3\sqrt{3}}{2}, -\frac{3}{2} + 4\sqrt{3})$
19	$(-7, 4)$
20	$(\frac{3}{2} + 5\sqrt{3}, 5 - \frac{3\sqrt{3}}{2})$

Solutions

1. Find the rank of $A = \begin{bmatrix} 5 & -2 \\ 6 & -5 \end{bmatrix}$. [1]

$\det A = (5)(-5) - (-2)(6) = -13$

$\det A \neq 0$, so A itself is a non-singular submatrix of order 2.

$\text{rank}(A) = 2$

Ans: 2

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2. Find the rank of $A = \begin{bmatrix} 1 & -1 & -1 \\ -3 & 3 & -3 \\ 2 & -2 & 4 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2]

$|A| = (12 - 6) + (-12 + 6) - (6 - 6) = 0$

$|A| = 0$, so the rank is less than 3.

The submatrix $\begin{bmatrix} 1 & -1 \\ -3 & -3 \end{bmatrix}$ has determinant $-6 \neq 0$.

$\therefore \text{rank}(A) = 2$

Ans: 2

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3. Find the rank of the matrix $\begin{bmatrix} 0 & 4 & 5 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. [1]

The non-zero rows begin with 1, 2, 3 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 3$

Ans: 3

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4. Reduce $A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ -2 & 1 & 0 & 4 \\ 1 & 2 & -1 & 3 \end{bmatrix}$ to echelon form and find its rank. [2]

Apply $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$: $A \sim \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 5 & -6 & 8 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

This is in echelon form with 3 non-zero rows.

$\therefore \text{rank}(A) = 3$

Ans: 3

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5. A and B are square matrices of order 7 with $\text{rank}(A) = 1$ and $\text{rank}(B) = 2$. What is the greatest possible value of $\text{rank}(A + B)$? [1]

$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

$1 + 2 = 3$

Ans: 3

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6. Write down the system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} 6 & 3 & -5 & 0 \\ -2 & 0 & -5 & -5 \\ 5 & 0 & 3 & 12 \end{array}\right]$. [1]

Each row gives one equation: the first three entries are the coefficients of x, y, z, the last is the constant.

$6x + 3y - 5z = 0$, $-2x - 5z = -5$ and $5x + 3z = 12$

Ans: $6x + 3y - 5z = 0$, $-2x - 5z = -5$ **and** $5x + 3z = 12$

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7. Show that the system $-6x + 2y = 8$ and $7x - 4y = -11$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{-6}{7} \neq \frac{b_1}{b_2} = \frac{2}{-4}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{10}\begin{bmatrix} -4 & -2 \\ -7 & -6 \end{bmatrix}\begin{bmatrix} 8 \\ -11 \end{bmatrix}$

$x = -1,\ y = 1$

Ans: $x = -1,\ y = 1$

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8. How many solutions does the system $5x - y = 6$ and $-15x + 3y = -13$ have? [2]

$\frac{a_1}{a_2} = \frac{5}{-15},\ \frac{b_1}{b_2} = \frac{-1}{3},\ \frac{c_1}{c_2} = \frac{6}{-13}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: the lines are parallel.

The system is inconsistent: no solution.

Ans: Inconsistent: no solution

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9. Show that the system $2x - 3y = 0$, $3x + 6y = 0$ has only the trivial solution. [2]

$\frac{a_1}{a_2} = \frac{2}{3},\ \frac{b_1}{b_2} = \frac{-3}{6}$

These are not equal (equivalently $|A| = 21 \neq 0$).

So the only solution is $x = y = 0$.

Ans: Only $x = 0$, $y = 0$

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10. Examine whether the homogeneous system $x - y + 6z = 0$, $2x + 5y - 9z = 0$ and $-5x - 4y - 3z = 0$ has only the trivial solution. [3]

$|A| = (-15 - 36) + (-6 - 45) + 6(-8 + 25) = 0$

$|A| = 0$, so the system has infinitely many solutions, including non-trivial ones.

Ans: Yes: $|A| = 0$**, so it has non-trivial solutions**

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11. Show that the system $2x + 3y + 2z = 0$, $x - 3y - 2z = 0$ and $-3x + 6y + 4z = 0$ has a non-trivial solution, and find all its solutions. [5]

$|A| = 2(-12 + 12) - 3(4 - 6) + 2(6 - 9) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = 3k$ in the first two equations:

$2x + 3y = -6k$, $x - 3y = 6k$

Solving (determinant $-9 \neq 0$): $x = 0,\ y = -2k$

These values also satisfy the third equation.

$(x, y, z) = (0, -2k, 3k)$, k any number

Ans: $(x, y, z) = (0, -2k, 3k)$**,** k **any number**

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12. Examine the consistency of $x + 4z = -3$, $x + 2y - 2z = -6$ and $-y + 3z = -2$ by the rank method. [3]

Apply $R_2 \to R_2 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & -3 \\ 0 & 2 & -6 & -3 \\ 0 & -1 & 3 & -2 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & -3 \\ 0 & -1 & 3 & -2 \\ 0 & 2 & -6 & -3 \end{array}\right]$

Apply $R_3 \to R_3 + 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & -3 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & -7 \end{array}\right]$

$\mathrm{rank}(A) = 2$ (non-zero rows of the coefficient part), $\mathrm{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\mathrm{rank}(A) = 2 \neq \mathrm{rank}[A|B] = 3$

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13. Examine the consistency of the system $x - z = 2$, $4x + 2y - z = 9$ and $-3z = 3$ by the rank method and solve it if consistent. [5]

Apply $R_2 \to R_2 - 4R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -3 & 3 \end{array}\right]$

$\mathrm{rank}(A) = \mathrm{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $-3z = 3$ gives $z = -1$

From row 2: $2y + 3z = 1$, so $y = 2$

From row 1: $x - z = 2$, so $x = 1$

$x = 1,\ y = 2,\ z = -1$

Ans: $x = 1,\ y = 2,\ z = -1$

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14. Solve $x - y - 3z = -12$, $-y - z = -6$ and $x - 2y - 4z = -18$ by the rank method. [5]

Apply $R_3 \to R_3 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -1 & -3 & -12 \\ 0 & -1 & -1 & -6 \\ 0 & -1 & -1 & -6 \end{array}\right]$

Apply $R_3 \to R_3 - R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -1 & -3 & -12 \\ 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\mathrm{rank}(A) = \mathrm{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $-y - z = -6$, so $y = 6 - k$

From row 1: $x - y - 3z = -12$, so $x = 2k - 6$

$(x, y, z) = (2k - 6, 6 - k, k)$, k any number

Ans: $(x, y, z) = (2k - 6, 6 - k, k)$, k **any number**

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15. Solve $x + 2y + z = 2$, $-2x - 3y - z = -4$ and $-3y = -3$ by Cramer's rule. [5]

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ -2 & -3 & -1 \\ 0 & -3 & 0 \end{vmatrix} = (0-3) - 2(0-0) + (6-0) = 3$$

$$\Delta_1 = \begin{vmatrix} 2 & 2 & 1 \\ -4 & -3 & -1 \\ -3 & -3 & 0 \end{vmatrix} = 2(0-3) - 2(0-3) + (12-9) = 3$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -1 \\ 0 & -3 & 0 \end{vmatrix} = (0-3) - 2(0-0) + (6-0) = 3$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ 0 & -3 & -3 \end{vmatrix} = (9-12) - 2(6-0) + 2(6-0) = -3$$

$x = \frac{\Delta_1}{\Delta} = \frac{3}{3} = 1,\ y = \frac{\Delta_2}{\Delta} = \frac{3}{3} = 1,\ z = \frac{\Delta_3}{\Delta} = \frac{-3}{3} = -1$

Ans: $x = 1,\ y = 1,\ z = -1$

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16. The point $(-2, 7)$ is reflected in the origin. Find its image. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

Ans: $(2, -7)$

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17. Under reflection in the x-axis, the image of a point P is $(-1, 4)$. Find P. [2]

Reflecting twice brings a point back, so reflect the image again.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$P = (-1, -4)$

Ans: $P(-1, -4)$

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18. The point $(8, -3)$ is reflected in the line $y = \sqrt{3}x$. Find the coordinates of its image. [3]

$\tan\theta = m = \sqrt{3}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = -\frac{1}{2},\ \sin 2\theta = \frac{2m}{1+m^2} = \frac{\sqrt{3}}{2}$

Reflection matrix: $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 - \frac{3\sqrt{3}}{2} \\ -\frac{3}{2} + 4\sqrt{3} \end{bmatrix}$$

Image: $(-4 - \frac{3\sqrt{3}}{2}, -\frac{3}{2} + 4\sqrt{3})$

Ans: $(-4 - \frac{3\sqrt{3}}{2}, -\frac{3}{2} + 4\sqrt{3})$

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19. Find the image of $(7, -4)$ when it is rotated anticlockwise about the origin through $180°$. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

Ans: $(-7, 4)$

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20. The point $(10, -3)$ is rotated anticlockwise about the origin through $30°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 10 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + 5\sqrt{3} \\ 5 - \frac{3\sqrt{3}}{2} \end{bmatrix}$

Image: $(\frac{3}{2} + 5\sqrt{3}, 5 - \frac{3\sqrt{3}}{2})$

Ans: $(\frac{3}{2} + 5\sqrt{3}, 5 - \frac{3\sqrt{3}}{2})$