EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 05

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. What is the rank of the matrix $\begin{bmatrix} -5 & -20 \\ -5 & -20 \end{bmatrix}$? [1 mark]

2. Find the rank of $A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & -2 & -2 \\ 4 & 6 & 4 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2 marks]

3. The matrix $\begin{bmatrix} -2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1 mark]

4. Find the rank of $A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ -2 & 2 & -1 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix}$ using elementary row transformations. [2 marks]

5. If $\text{rank}(A) = 2$, $\text{rank}(B) = 1$ and AB is defined, what is the greatest possible value of $\text{rank}(AB)$? [1 mark]

6. Write the augmented matrix of the system $3x + 5y + 3z = -2$, $-2x + 4z = -5$ and $5x - y - z = 10$. [1 mark]

7. For what value of k does the system $kx + 3y = 3$, $-3x - 9y = -12$ have no solution? [2 marks]

8. How many solutions does the system $-6x + 4y = 8$ and $-12x + 8y = 16$ have? [2 marks]

9. Show that the system $-5x + 4y = 0$, $-6x - y = 0$ has only the trivial solution. [2 marks]

10. Does the system $y - z = 0$, $-3x + y - 2z = 0$ and $-3x - 3y + 2z = 0$ have a non-trivial solution? [3 marks]

11. Show that the system $x + y - 2z = 0$, $x + 7y - 5z = 0$ and $4x - 2y - 5z = 0$ has a non-trivial solution, and find all its solutions. [5 marks]

12. Examine the consistency of $x - 2y + z = -6$, $2x - 3y + 3z = -6$ and $-4x + 7y - 5z = 1$ by the rank method. [3 marks]

13. Examine the consistency of the system $x - 3y = 5$, $3x - 3y + 3z = 6$ and $-2y = 4$ by the rank method and solve it if consistent. [5 marks]

14. Show that the system $x + 2y + z = 7$, $x + 3y + z = 11$ and $2x + 5y + 2z = 18$ is consistent and has infinitely many solutions. Find them. [5 marks]

15. Using Cramer's rule, solve the system $-3x + 3y + 3z = -12$, $-3x + 2y + 2z = -9$ and $4x + y + 2z = -2$. [5 marks]

16. The point $(-2, -3)$ is reflected in the origin. Find its image. [1 mark]

17. The point $(10, -9)$ is reflected in the line $y = x$, and the image is then reflected in the line $y = -x$. Find the final image. [2 marks]

18. Find the image of $(-5, -3)$ under reflection in the line $y = \frac{3}{4}x$. [3 marks]

19. Find the image of $(-1, -6)$ when it is rotated anticlockwise about the origin through $90°$. [1 mark]

20. The point $(-1, -6)$ is rotated anticlockwise about the origin through $60°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	1
2	2
3	2
4	3
5	1
6	$\left[\begin{array}{ccc\|c} 3 & 5 & 3 & -2 \\ -2 & 0 & 4 & -5 \\ 5 & -1 & -1 & 10 \end{array}\right]$
7	$k = 1$
8	Consistent: infinitely many solutions
9	Only $x = 0$, $y = 0$
10	Yes: $\|A\| = 0$, so it has non-trivial solutions
11	$(x, y, z) = (3k, k, 2k)$, k any number
12	Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A\|B] = 3$
13	$x = -1,\ y = -2,\ z = 1$
14	$(x, y, z) = (-k - 1, 4, k)$, k any number
15	$x = 1,\ y = 0,\ z = -3$
16	$(2, 3)$
17	$(-10, 9)$
18	$(-\frac{107}{25}, -\frac{99}{25})$
19	$(6, -1)$
20	$(-\frac{1}{2} + 3\sqrt{3}, -3 - \frac{\sqrt{3}}{2})$

Solutions

1. What is the rank of the matrix $\begin{bmatrix} -5 & -20 \\ -5 & -20 \end{bmatrix}$? [1]

$\det A = (-5)(-20) - (-20)(-5) = 0$

A is singular but not null, and any non-zero element is a non-singular submatrix of order 1.

$\text{rank}(A) = 1$

Ans: 1

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2. Find the rank of $A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & -2 & -2 \\ 4 & 6 & 4 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2]

$|A| = (-8 + 12) + (-12 + 8) + 0(-18 + 8) = 0$

$|A| = 0$, so the rank is less than 3.

The submatrix $\begin{bmatrix} 1 & -1 \\ -3 & -2 \end{bmatrix}$ has determinant $-5 \neq 0$.

$\therefore \text{rank}(A) = 2$

Ans: 2

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3. The matrix $\begin{bmatrix} -2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1]

The non-zero rows begin with 0, 3 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 2$

Ans: 2

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4. Find the rank of $A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ -2 & 2 & -1 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix}$ using elementary row transformations. [2]

Apply $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + 3R_1$: $A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 3 & -3 \\ 0 & 0 & 7 & -6 \end{bmatrix}$

This is in echelon form with 3 non-zero rows.

$\therefore \text{rank}(A) = 3$

Ans: 3

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5. If $\text{rank}(A) = 2$, $\text{rank}(B) = 1$ and AB is defined, what is the greatest possible value of $\text{rank}(AB)$? [1]

$\text{rank}(AB)$ cannot exceed the rank of either A or B.

$\text{rank}(AB) \leq \min(2, 1) = 1$

Ans: 1

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6. Write the augmented matrix of the system $3x + 5y + 3z = -2$, $-2x + 4z = -5$ and $5x - y - z = 10$. [1]

Coefficient matrix $A = \begin{bmatrix} 3 & 5 & 3 \\ -2 & 0 & 4 \\ 5 & -1 & -1 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} -2 \\ -5 \\ 10 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} 3 & 5 & 3 & -2 \\ -2 & 0 & 4 & -5 \\ 5 & -1 & -1 & 10 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} 3 & 5 & 3 & -2 \\ -2 & 0 & 4 & -5 \\ 5 & -1 & -1 & 10 \end{array}\right]$

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7. For what value of k does the system $kx + 3y = 3$, $-3x - 9y = -12$ have no solution? [2]

No solution needs $\frac{k}{-3} = \frac{3}{-9} \neq \frac{3}{-12}$

$\frac{k}{-3} = \frac{1}{-3} \Rightarrow k = 1$

With $k = 1$ the constants are not in the same ratio, so the lines are parallel.

Ans: $k = 1$

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8. How many solutions does the system $-6x + 4y = 8$ and $-12x + 8y = 16$ have? [2]

$\frac{a_1}{a_2} = \frac{-6}{-12}$, $\frac{b_1}{b_2} = \frac{4}{8}$, $\frac{c_1}{c_2} = \frac{8}{16}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: the lines coincide.

The system is consistent with infinitely many solutions.

Ans: Consistent: infinitely many solutions

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9. Show that the system $-5x + 4y = 0$, $-6x - y = 0$ has only the trivial solution. [2]

$\frac{a_1}{a_2} = \frac{-5}{-6}$, $\frac{b_1}{b_2} = \frac{4}{-1}$

These are not equal (equivalently $|A| = 29 \neq 0$).

So the only solution is $x = y = 0$.

Ans: Only $x = 0$, $y = 0$

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10. Does the system $y - z = 0$, $-3x + y - 2z = 0$ and $-3x - 3y + 2z = 0$ have a non-trivial solution? [3]

$|A| = 0(2 - 6) - (-6 - 6) - (9 + 3) = 0$

$|A| = 0$, so the system has infinitely many solutions, including non-trivial ones.

Ans: Yes: $|A| = 0$, so it has non-trivial solutions

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11. Show that the system $x + y - 2z = 0$, $x + 7y - 5z = 0$ and $4x - 2y - 5z = 0$ has a non-trivial solution, and find all its solutions. [5]

$|A| = (-35 - 10) - (-5 + 20) - 2(-2 - 28) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = 2k$ in the first two equations:

$x + y = 4k$, $x + 7y = 10k$

Solving (determinant $6 \neq 0$): $x = 3k,\ y = k$

These values also satisfy the third equation.

$(x, y, z) = (3k, k, 2k)$, k any number

Ans: $(x, y, z) = (3k, k, 2k)$, k **any number**

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12. Examine the consistency of $x - 2y + z = -6$, $2x - 3y + 3z = -6$ and $-4x + 7y - 5z = 1$ by the rank method. [3]

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 4R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & -6 \\ 0 & 1 & 1 & 6 \\ 0 & -1 & -1 & -23 \end{array}\right]$

Apply $R_3 \to R_3 + R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & -6 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & -17 \end{array}\right]$

$\text{rank}(A) = 2$ (non-zero rows of the coefficient part), $\text{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A|B] = 3$

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13. Examine the consistency of the system $x - 3y = 5$, $3x - 3y + 3z = 6$ and $-2y = 4$ by the rank method and solve it if consistent. [5]

Apply $R_2 \to R_2 - 3R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 6 & 3 & -9 \\ 0 & -2 & 0 & 4 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 0 & 4 \\ 0 & 6 & 3 & -9 \end{array}\right]$

Apply $R_3 \to R_3 + 3R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 3 & 3 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $3z = 3$ gives $z = 1$

From row 2: $-2y = 4$, so $y = -2$

From row 1: $x - 3y = 5$, so $x = -1$

$x = -1,\ y = -2,\ z = 1$

Ans: $x = -1,\ y = -2,\ z = 1$

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14. Show that the system $x + 2y + z = 7$, $x + 3y + z = 11$ and $2x + 5y + 2z = 18$ is consistent and has infinitely many solutions. Find them. [5]

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 4 \end{array}\right]$

Apply $R_3 \to R_3 - R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $y = 4$, so $y = 4$

From row 1: $x + 2y + z = 7$, so $x = -k - 1$

$(x, y, z) = (-k - 1, 4, k)$, k any number

Ans: $(x, y, z) = (-k - 1, 4, k)$, k **any number**

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15. Using Cramer's rule, solve the system $-3x + 3y + 3z = -12$, $-3x + 2y + 2z = -9$ and $4x + y + 2z = -2$. [5]

$$\Delta = \begin{vmatrix} -3 & 3 & 3 \\ -3 & 2 & 2 \\ 4 & 1 & 2 \end{vmatrix} = -3(4-2) - 3(-6-8) + 3(-3-8) = 3$$

$$\Delta_1 = \begin{vmatrix} -12 & 3 & 3 \\ -9 & 2 & 2 \\ -2 & 1 & 2 \end{vmatrix} = -12(4-2) - 3(-18+4) + 3(-9+4) = 3$$

$$\Delta_2 = \begin{vmatrix} -3 & -12 & 3 \\ -3 & -9 & 2 \\ 4 & -2 & 2 \end{vmatrix} = -3(-18+4) + 12(-6-8) + 3(6+36) = 0$$

$$\Delta_3 = \begin{vmatrix} -3 & 3 & -12 \\ -3 & 2 & -9 \\ 4 & 1 & -2 \end{vmatrix} = -3(-4+9) - 3(6+36) - 12(-3-8) = -9$$

$x = \frac{\Delta_1}{\Delta} = \frac{3}{3} = 1,\ y = \frac{\Delta_2}{\Delta} = \frac{0}{3} = 0,\ z = \frac{\Delta_3}{\Delta} = \frac{-9}{3} = -3$

Ans: $x = 1,\ y = 0,\ z = -3$

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16. The point $(-2, -3)$ is reflected in the origin. Find its image. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Ans: $(2, 3)$

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17. The point $(10, -9)$ is reflected in the line $y = x$, and the image is then reflected in the line $y = -x$. Find the final image. [2]

Reflection in the line $y = x$: $(10, -9) \mapsto (-9, 10)$

Reflection in the line $y = -x$: $(-9, 10) \mapsto (-10, 9)$

In matrix form: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ -9 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \end{bmatrix}$

Ans: $(-10, 9)$

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18. Find the image of $(-5, -3)$ under reflection in the line $y = \frac{3}{4}x$. [3]

$\tan\theta = m = \frac{3}{4}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = \frac{7}{25}$, $\sin 2\theta = \frac{2m}{1+m^2} = \frac{24}{25}$

Reflection matrix: $\begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & -\frac{7}{25} \end{bmatrix}$

$$\begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & -\frac{7}{25} \end{bmatrix} \begin{bmatrix} -5 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{107}{25} \\ -\frac{99}{25} \end{bmatrix}$$

Image: $(-\frac{107}{25}, -\frac{99}{25})$

Ans: $(-\frac{107}{25}, -\frac{99}{25})$

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19. Find the image of $(-1, -6)$ when it is rotated anticlockwise about the origin through $90°$. [1]

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

Ans: $(6, -1)$

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20. The point $(-1, -6)$ is rotated anticlockwise about the origin through $60°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + 3\sqrt{3} \\ -3 - \frac{\sqrt{3}}{2} \end{bmatrix}$$

Image: $(-\frac{1}{2} + 3\sqrt{3}, -3 - \frac{\sqrt{3}}{2})$

Ans: $(-\frac{1}{2} + 3\sqrt{3}, -3 - \frac{\sqrt{3}}{2})$
