EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 6 & 2 \\ 18 & 6 \end{bmatrix}$. [1 mark]

2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 0 \\ -3 & 3 & -3 \end{bmatrix}$. [2 marks]

3. The matrix $\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1 mark]

4. Reduce $A = \begin{bmatrix} 1 & -2 & -3 & 4 \\ -3 & -6 & -1 & -2 \\ 1 & 4 & 2 & -1 \end{bmatrix}$ to echelon form and find its rank. [2 marks]

5. What is the rank of the 3×5 null matrix? [1 mark]

6. Write the augmented matrix of the system $-2x + 3y + 4z = 0$, $5x + 6y + 3z = 2$ and $-5x = 9$. [1 mark]

7. Show that the system $7x + 3y = 50$ and $-5x - 3y = -40$ has a unique solution, and find it. [2 marks]

8. Show that the system $2x - 4y = -14$ and $6x - y = -20$ has a unique solution, and find it. [2 marks]

9. Show that the system $4x - 2y = 0$, $-5x + 7y = 0$ has only the trivial solution. [2 marks]

10. Examine whether the homogeneous system $-3x - y - 7z = 0$, $-x - y - z = 0$ and $6x + 5y + 8z = 0$ has only the trivial solution. [3 marks]

11. Solve the homogeneous system $2x - y + 8z = 0$, $-x - 4y + 5z = 0$ and $-6x - 5y - 8z = 0$. [5 marks]

12. Examine the consistency of $x - 3y = 3$, $x - 6y + 4z = 4$ and $-3y + 4z = -3$ by the rank method. [3 marks]

13. Examine the consistency of the system $x + 4y + 4z = 18$, $3x - 3y - 2z = -5$ and $y + 2z = 5$ by the rank method and solve it if consistent. [5 marks]

14. Solve $x + y - z = -3$, $-2x - y - 2z = 4$ and $y - 4z = -2$ by the rank method. [5 marks]

15. Using Cramer's rule, solve the system $-2x + y + z = -3$, $-2x - 3y - z = -17$ and $3x - 2y - 2z = 3$. [5 marks]

16. The point $(-1, 3)$ is reflected in the origin. Find its image. [1 mark]

17. The point $(-2, 10)$ is reflected in the origin, and the image is then reflected in the x-axis. Find the final image. [2 marks]

18. The point $(7, 1)$ is reflected in the line $y = \frac{2}{3}x$. Find the coordinates of its image. [3 marks]

19. Find the image of $(5, -1)$ when it is rotated anticlockwise about the origin through $180°$. [1 mark]

20. The point $(-6, -1)$ is rotated anticlockwise about the origin through $120°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	1
2	3
3	2
4	2
5	0
6	$\left[\begin{array}{ccc\|c} -2 & 3 & 4 & 0 \\ 5 & 6 & 3 & 2 \\ -5 & 0 & 0 & 9 \end{array}\right]$
7	$x = 5,\ y = 5$
8	$x = -3,\ y = 2$
9	Only $x = 0, y = 0$
10	Yes: $\lvert A\rvert = 0$, so it has non-trivial solutions
11	$(x, y, z) = (-3k, 2k, k)$, k any number
12	Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A\lvert B] = 3$
13	$x = 2,\ y = 3,\ z = 1$
14	$(x, y, z) = (-3k - 1, 4k - 2, k)$, k any number
15	$x = 3,\ y = 4,\ z = -1$
16	$(1, -3)$
17	$(2, 10)$
18	$(\frac{47}{13}, \frac{79}{13})$
19	$(-5, 1)$
20	$(\frac{\sqrt{3}}{2} + 3, \frac{1}{2} - 3\sqrt{3})$

Solutions

1. Find the rank of $A = \begin{bmatrix} 6 & 2 \\ 18 & 6 \end{bmatrix}$. [1]

$\det A = (6)(6) - (2)(18) = 0$

A is singular but not null, and any non-zero element is a non-singular submatrix of order 1.

$\text{rank}(A) = 1$

Ans: 1

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2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 0 \\ -3 & 3 & -3 \end{bmatrix}$. [2]

$|A| = 0(0-0) + 2(-3-0) - (3-0) = -9$

$|A| \neq 0$, so A is non-singular.

$\text{rank}(A) = 3$

Ans: 3

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3. The matrix $\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1]

The non-zero rows begin with 1, 2 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 2$

Ans: 2

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4. Reduce $A = \begin{bmatrix} 1 & -2 & -3 & 4 \\ -3 & -6 & -1 & -2 \\ 1 & 4 & 2 & -1 \end{bmatrix}$ to echelon form and find its rank. [2]

Apply $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - R_1$: $A \sim \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & -12 & -10 & 10 \\ 0 & 6 & 5 & -5 \end{bmatrix}$

Apply $R_2 \leftrightarrow R_3$: $\sim \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 6 & 5 & -5 \\ 0 & -12 & -10 & 10 \end{bmatrix}$

Apply $R_3 \to R_3 + 2R_2$: $\sim \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 6 & 5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in echelon form with 2 non-zero rows.

$\therefore \text{rank}(A) = 2$

Ans: 2

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5. What is the rank of the 3×5 null matrix? [1]

The rank of a null matrix is defined to be 0.

Ans: 0

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6. Write the augmented matrix of the system $-2x + 3y + 4z = 0$, $5x + 6y + 3z = 2$ and $-5x = 9$. [1]

Coefficient matrix $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & 6 & 3 \\ -5 & 0 & 0 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} 0 \\ 2 \\ 9 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} -2 & 3 & 4 & 0 \\ 5 & 6 & 3 & 2 \\ -5 & 0 & 0 & 9 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} -2 & 3 & 4 & 0 \\ 5 & 6 & 3 & 2 \\ -5 & 0 & 0 & 9 \end{array}\right]$

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7. Show that the system $7x + 3y = 50$ and $-5x - 3y = -40$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{7}{-5} \neq \frac{b_1}{b_2} = \frac{3}{-3}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{-6}\begin{bmatrix} -3 & -3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 50 \\ -40 \end{bmatrix}$

$x = 5,\ y = 5$

Ans: $x = 5,\ y = 5$

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8. Show that the system $2x - 4y = -14$ and $6x - y = -20$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{2}{6} \neq \frac{b_1}{b_2} = \frac{-4}{-1}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{22}\begin{bmatrix} -1 & 4 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -14 \\ -20 \end{bmatrix}$

$x = -3,\ y = 2$

Ans: $x = -3,\ y = 2$

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9. Show that the system $4x - 2y = 0$, $-5x + 7y = 0$ has only the trivial solution. [2]

$\frac{a_1}{a_2} = \frac{4}{-5},\ \frac{b_1}{b_2} = \frac{-2}{7}$

These are not equal (equivalently $|A| = 18 \neq 0$).

So the only solution is $x = y = 0$.

Ans: Only $x = 0, y = 0$

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10. Examine whether the homogeneous system $-3x - y - 7z = 0$, $-x - y - z = 0$ and $6x + 5y + 8z = 0$ has only the trivial solution. [3]

$|A| = -3(-8 + 5) + (-8 + 6) - 7(-5 + 6) = 0$

$|A| = 0$, so the system has infinitely many solutions, including non-trivial ones.

Ans: Yes: $|A| = 0$**, so it has non-trivial solutions**

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11. Solve the homogeneous system $2x - y + 8z = 0$, $-x - 4y + 5z = 0$ and $-6x - 5y - 8z = 0$. [5]

$|A| = 2(32 + 25) + (8 + 30) + 8(5 - 24) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = k$ in the first two equations:

$2x - y = -8k$, $-x - 4y = -5k$

Solving (determinant $-9 \neq 0$): $x = -3k,\ y = 2k$

These values also satisfy the third equation.

$(x, y, z) = (-3k, 2k, k)$, k any number

Ans: $(x, y, z) = (-3k, 2k, k)$, k **any number**

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12. Examine the consistency of $x - 3y = 3$, $x - 6y + 4z = 4$ and $-3y + 4z = -3$ by the rank method. [3]

Apply $R_2 \to R_2 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 3 \\ 0 & -3 & 4 & 1 \\ 0 & -3 & 4 & -3 \end{array}\right]$

Apply $R_3 \to R_3 - R_2$: $\sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 3 \\ 0 & -3 & 4 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$

$\text{rank}(A) = 2$ (non-zero rows of the coefficient part), $\text{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A|B] = 3$

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13. Examine the consistency of the system $x + 4y + 4z = 18$, $3x - 3y - 2z = -5$ and $y + 2z = 5$ by the rank method and solve it if consistent. [5]

Apply $R_2 \to R_2 - 3R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 4 & 4 & 18 \\ 0 & -15 & -14 & -59 \\ 0 & 1 & 2 & 5 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & 4 & 4 & 18 \\ 0 & 1 & 2 & 5 \\ 0 & -15 & -14 & -59 \end{array}\right]$

Apply $R_3 \to R_3 + 15R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 4 & 4 & 18 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 16 & 16 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $16z = 16$ gives $z = 1$

From row 2: $y + 2z = 5$, so $y = 3$

From row 1: $x + 4y + 4z = 18$, so $x = 2$

$x = 2,\ y = 3,\ z = 1$

Ans: $x = 2,\ y = 3,\ z = 1$

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14. Solve $x + y - z = -3$, $-2x - y - 2z = 4$ and $y - 4z = -2$ by the rank method. [5]

Apply $R_2 \to R_2 + 2R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & -4 & -2 \\ 0 & 1 & -4 & -2 \end{array}\right]$

Apply $R_3 \to R_3 - R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $y - 4z = -2$, so $y = 4k - 2$

From row 1: $x + y - z = -3$, so $x = -3k - 1$

$(x, y, z) = (-3k - 1, 4k - 2, k)$, k any number

Ans: $(x, y, z) = (-3k - 1, 4k - 2, k)$, k **any number**

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15. Using Cramer's rule, solve the system $-2x + y + z = -3$, $-2x - 3y - z = -17$ and $3x - 2y - 2z = 3$. [5]

$$\Delta = \begin{vmatrix} -2 & 1 & 1 \\ -2 & -3 & -1 \\ 3 & -2 & -2 \end{vmatrix} = -2(6 - 2) - (4 + 3) + (4 + 9) = -2$$

$$\Delta_1 = \begin{vmatrix} -3 & 1 & 1 \\ -17 & -3 & -1 \\ 3 & -2 & -2 \end{vmatrix} = -3(6 - 2) - (34 + 3) + (34 + 9) = -6$$

$$\Delta_2 = \begin{vmatrix} -2 & -3 & 1 \\ -2 & -17 & -1 \\ 3 & 3 & -2 \end{vmatrix} = -2(34 + 3) + 3(4 + 3) + (-6 + 51) = -8$$

$$\Delta_3 = \begin{vmatrix} -2 & 1 & -3 \\ -2 & -3 & -17 \\ 3 & -2 & 3 \end{vmatrix} = -2(-9 - 34) - (-6 + 51) - 3(4 + 9) = 2$$

$x = \frac{\Delta_1}{\Delta} = \frac{-6}{-2} = 3,\ y = \frac{\Delta_2}{\Delta} = \frac{-8}{-2} = 4,\ z = \frac{\Delta_3}{\Delta} = \frac{2}{-2} = -1$

Ans: $x = 3,\ y = 4,\ z = -1$

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16. The point $(-1, 3)$ is reflected in the origin. Find its image. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Ans: $(1, -3)$

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17. The point $(-2, 10)$ is reflected in the origin, and the image is then reflected in the x-axis. Find the final image. [2]

Reflection in the origin: $(-2, 10) \mapsto (2, -10)$

Reflection in the x-axis: $(2, -10) \mapsto (2, 10)$

In matrix form: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

Ans: $(2, 10)$

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18. The point $(7, 1)$ is reflected in the line $y = \frac{2}{3}x$. Find the coordinates of its image. [3]

$\tan\theta = m = \frac{2}{3}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = \frac{5}{13}$, $\sin 2\theta = \frac{2m}{1+m^2} = \frac{12}{13}$

Reflection matrix: $\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$

$$\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{47}{13} \\ \frac{79}{13} \end{bmatrix}$$

Image: $(\frac{47}{13}, \frac{79}{13})$

Ans: $(\frac{47}{13}, \frac{79}{13})$

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19. Find the image of $(5, -1)$ when it is rotated anticlockwise about the origin through $180°$. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

Ans: $(-5, 1)$

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20. The point $(-6, -1)$ is rotated anticlockwise about the origin through $120°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -6 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} + 3 \\ \frac{1}{2} - 3\sqrt{3} \end{bmatrix}$$

Image: $(\frac{\sqrt{3}}{2} + 3, \frac{1}{2} - 3\sqrt{3})$

Ans: $(\frac{\sqrt{3}}{2} + 3, \frac{1}{2} - 3\sqrt{3})$
