EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 03

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix}$. [1 mark]

2. Find the rank of $A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & 2 & -3 \\ 3 & -8 & 4 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2 marks]

3. The matrix $\begin{bmatrix} 0 & 2 & 7 & -4 & 6 \\ 0 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$ is in echelon form. Write its rank. [1 mark]

4. Find the rank of $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 3 & 1 & 3 \\ -3 & 1 & 1 & 0 \end{bmatrix}$ using elementary row transformations. [2 marks]

5. A and B are square matrices of order 5 with $\text{rank}(A) = 2$ and $\text{rank}(B) = 3$. What is the greatest possible value of $\text{rank}(A + B)$? [1 mark]

6. Write the augmented matrix of the system $x - y = -3$, $6x - 4y - 2z = 11$ and $6x + 5y - z = 2$. [1 mark]

7. How many solutions does the system $-5x - 3y = 8$ and $-10x - 6y = 16$ have? [2 marks]

8. Examine whether the system $2x + y = 5$ and $-4x - 2y = -5$ is consistent. [2 marks]

9. Find k so that the system $7x + 8y = 0$, $kx + 40y = 0$ has a non-trivial solution. [2 marks]

10. Does the system $-2x - y - 3z = 0$, $-2x + 4y = 0$ and $3x - 2y + 3z = 0$ have a non-trivial solution? [3 marks]

11. Show that the system $x - y - 4z = 0$, $-3x - 4y - 2z = 0$ and $x - 2z = 0$ has a non-trivial solution, and find all its solutions. [5 marks]

12. Show that the system $x + 2y = 7$, $10y + 4z = -2$ and $x - 3y - 2z = 0$ is inconsistent, using the rank method. [3 marks]

13. Using the rank method, show that $x - 2y + 3z = -7$, $4x - 2y + 3z = -16$ and $-x + 2y - z = 3$ is consistent and find its solution. [5 marks]

14. Solve $x - z = -1$, $x - y = -5$ and $x + y - 2z = 3$ by the rank method. [5 marks]

15. Solve $2x - y - 2z = 5$, $4x + z = 2$ and $4x + 4z = -4$ by Cramer's rule. [5 marks]

16. The point $(-6, 1)$ is reflected in the y-axis. Find its image. [1 mark]

17. The point $(1, -5)$ is reflected in the line $y = -x$, and the image is then reflected in the origin. Find the final image. [2 marks]

18. The point $(-2, 3)$ is reflected in the line $y = \frac{1}{2}x$. Find the coordinates of its image. [3 marks]

19. Find the image of $(6, -4)$ when it is rotated anticlockwise about the origin through $180°$. [1 mark]

20. The point $(9, -5)$ is rotated anticlockwise about the origin through $135°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	2
2	2
3	3
4	3
5	5
6	$\left[\begin{array}{ccc\|c} 1 & -1 & 0 & -3 \\ 6 & -4 & -2 & 11 \\ 6 & 5 & -1 & 2 \end{array}\right]$
7	Consistent: infinitely many solutions
8	Inconsistent: no solution
9	$k = 35$
10	No: only the trivial solution ($\lvert A\rvert = -6 \neq 0$)
11	$(x, y, z) = (-2k, 2k, -k)$, k any number
12	Inconsistent: $\mathrm{rank}(A) = 2 \neq \mathrm{rank}[A\lvert B] = 3$
13	$x = -3,\ y = -1,\ z = -2$
14	$(x, y, z) = (k - 1, k + 4, k)$, k any number
15	$x = 1,\ y = 1,\ z = -2$
16	$(6, 1)$
17	$(-5, 1)$
18	$(\frac{6}{5}, -\frac{17}{5})$
19	$(-6, 4)$
20	$(-2\sqrt{2}, 7\sqrt{2})$

Solutions

1. Find the rank of $A = \begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix}$. [1]

$\det A = (2)(1) - (-5)(1) = 7$

$\det A \neq 0$, so A itself is a non-singular submatrix of order 2.

$\text{rank}(A) = 2$

Ans: 2

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2. Find the rank of $A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & 2 & -3 \\ 3 & -8 & 4 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2]

$|A| = (8 - 24) - 4(-8 + 9) + 2(16 - 6) = 0$

$|A| = 0$, so the rank is less than 3.

The submatrix $\begin{bmatrix} 1 & 4 \\ -2 & 2 \end{bmatrix}$ has determinant $10 \neq 0$.

$\therefore \text{rank}(A) = 2$

Ans: 2

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3. The matrix $\begin{bmatrix} 0 & 2 & 7 & -4 & 6 \\ 0 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$ is in echelon form. Write its rank. [1]

The non-zero rows begin with 1, 2, 4 zeros: the count increases row by row.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 3$

Ans: 3

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4. Find the rank of $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 3 & 1 & 3 \\ -3 & 1 & 1 & 0 \end{bmatrix}$ using elementary row transformations. [2]

Apply $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 3R_1$: $A \sim \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 4 & -5 & 3 \end{bmatrix}$

Apply $R_2 \leftrightarrow R_3$: $\sim \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 4 & -5 & 3 \\ 0 & 0 & 7 & 0 \end{bmatrix}$

This is in echelon form with 3 non-zero rows.

$\therefore \text{rank}(A) = 3$

Ans: 3

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5. A and B are square matrices of order 5 with $\text{rank}(A) = 2$ and $\text{rank}(B) = 3$. What is the greatest possible value of $\text{rank}(A + B)$? [1]

$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

$2 + 3 = 5$

Ans: 5

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6. Write the augmented matrix of the system $x - y = -3$, $6x - 4y - 2z = 11$ and $6x + 5y - z = 2$. [1]

Coefficient matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 6 & -4 & -2 \\ 6 & 5 & -1 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} -3 \\ 11 \\ 2 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 6 & -4 & -2 & 11 \\ 6 & 5 & -1 & 2 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 6 & -4 & -2 & 11 \\ 6 & 5 & -1 & 2 \end{array}\right]$

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7. How many solutions does the system $-5x - 3y = 8$ and $-10x - 6y = 16$ have? [2]

$\frac{a_1}{a_2} = \frac{-5}{-10}$, $\frac{b_1}{b_2} = \frac{-3}{-6}$, $\frac{c_1}{c_2} = \frac{8}{16}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: the lines coincide.

The system is consistent with infinitely many solutions.

Ans: Consistent: infinitely many solutions

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8. Examine whether the system $2x + y = 5$ and $-4x - 2y = -5$ is consistent. [2]

$\frac{a_1}{a_2} = \frac{2}{-4}$, $\frac{b_1}{b_2} = \frac{1}{-2}$, $\frac{c_1}{c_2} = \frac{5}{-5}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: the lines are parallel.

The system is inconsistent: no solution.

Ans: Inconsistent: no solution

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9. Find k so that the system $7x + 8y = 0$, $kx + 40y = 0$ has a non-trivial solution. [2]

A non-trivial solution needs $\frac{7}{k} = \frac{8}{40}$

$k = 35$

Ans: $k = 35$

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10. Does the system $-2x - y - 3z = 0$, $-2x + 4y = 0$ and $3x - 2y + 3z = 0$ have a non-trivial solution? [3]

$|A| = -2(12 - 0) + (-6 - 0) - 3(4 - 12) = -6$

$|A| \neq 0$, so the system has only the trivial solution $x = y = z = 0$.

Ans: No: only the trivial solution ($|A| = -6 \neq 0$)

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11. Show that the system $x - y - 4z = 0$, $-3x - 4y - 2z = 0$ and $x - 2z = 0$ has a non-trivial solution, and find all its solutions. [5]

$|A| = (8 - 0) + (6 + 2) - 4(0 + 4) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = -k$ in the first two equations:

$x - y = -4k$, $-3x - 4y = -2k$

Solving (determinant $-7 \neq 0$): $x = -2k,\ y = 2k$

These values also satisfy the third equation.

$(x, y, z) = (-2k, 2k, -k)$, k any number

Ans: $(x, y, z) = (-2k, 2k, -k)$, k **any number**

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12. Show that the system $x + 2y = 7$, $10y + 4z = -2$ and $x - 3y - 2z = 0$ is inconsistent, using the rank method. [3]

Apply $R_3 \to R_3 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 10 & 4 & -2 \\ 0 & -5 & -2 & -7 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & -5 & -2 & -7 \\ 0 & 10 & 4 & -2 \end{array}\right]$

Apply $R_3 \to R_3 + 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & -5 & -2 & -7 \\ 0 & 0 & 0 & -16 \end{array}\right]$

$\mathrm{rank}(A) = 2$ (non-zero rows of the coefficient part), $\mathrm{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\mathrm{rank}(A) = 2 \neq \mathrm{rank}[A|B] = 3$

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13. Using the rank method, show that $x - 2y + 3z = -7$, $4x - 2y + 3z = -16$ and $-x + 2y - z = 3$ is consistent and find its solution. [5]

Apply $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 + R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -7 \\ 0 & 6 & -9 & 12 \\ 0 & 0 & 2 & -4 \end{array}\right]$

$\mathrm{rank}(A) = \mathrm{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $2z = -4$ gives $z = -2$

From row 2: $6y - 9z = 12$, so $y = -1$

From row 1: $x - 2y + 3z = -7$, so $x = -3$

$x = -3,\ y = -1,\ z = -2$

Ans: $x = -3,\ y = -1,\ z = -2$

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14. Solve $x - z = -1$, $x - y = -5$ and $x + y - 2z = 3$ by the rank method. [5]

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & -4 \\ 0 & 1 & -1 & 4 \end{array}\right]$

Apply $R_3 \to R_3 + R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $-y + z = -4$, so $y = k + 4$

From row 1: $x - z = -1$, so $x = k - 1$

$(x, y, z) = (k - 1, k + 4, k)$, k any number

Ans: $(x, y, z) = (k - 1, k + 4, k)$, k **any number**

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15. Solve $2x - y - 2z = 5$, $4x + z = 2$ and $4x + 4z = -4$ by Cramer's rule. [5]

$$\Delta = \begin{vmatrix} 2 & -1 & -2 \\ 4 & 0 & 1 \\ 4 & 0 & 4 \end{vmatrix} = 2(0 - 0) + (16 - 4) - 2(0 - 0) = 12$$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & -2 \\ 2 & 0 & 1 \\ -4 & 0 & 4 \end{vmatrix} = 5(0 - 0) + (8 + 4) - 2(0 - 0) = 12$$

$$\Delta_2 = \begin{vmatrix} 2 & 5 & -2 \\ 4 & 2 & 1 \\ 4 & -4 & 4 \end{vmatrix} = 2(8 + 4) - 5(16 - 4) - 2(-16 - 8) = 12$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 5 \\ 4 & 0 & 2 \\ 4 & 0 & -4 \end{vmatrix} = 2(0 - 0) + (-16 - 8) + 5(0 - 0) = -24$$

$x = \frac{\Delta_1}{\Delta} = \frac{12}{12} = 1,\ y = \frac{\Delta_2}{\Delta} = \frac{12}{12} = 1,\ z = \frac{\Delta_3}{\Delta} = \frac{-24}{12} = -2$

Ans: $x = 1,\ y = 1,\ z = -2$

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16. The point $(-6, 1)$ is reflected in the y-axis. Find its image. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Ans: $(6, 1)$

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17. The point $(1, -5)$ is reflected in the line $y = -x$, and the image is then reflected in the origin. Find the final image. [2]

Reflection in the line $y = -x$: $(1, -5) \mapsto (5, -1)$

Reflection in the origin: $(5, -1) \mapsto (-5, 1)$

In matrix form: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

Ans: $(-5, 1)$

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18. The point $(-2, 3)$ is reflected in the line $y = \frac{1}{2}x$. Find the coordinates of its image. [3]

$\tan\theta = m = \frac{1}{2}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = \frac{3}{5}$, $\sin 2\theta = \frac{2m}{1+m^2} = \frac{4}{5}$

Reflection matrix: $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{17}{5} \end{bmatrix}$$

Image: $(\frac{6}{5}, -\frac{17}{5})$

Ans: $(\frac{6}{5}, -\frac{17}{5})$

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19. Find the image of $(6, -4)$ when it is rotated anticlockwise about the origin through $180°$. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

Ans: $(-6, 4)$

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20. The point $(9, -5)$ is rotated anticlockwise about the origin through $135°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 9 \\ -5 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ 7\sqrt{2} \end{bmatrix}$$

Image: $(-2\sqrt{2}, 7\sqrt{2})$

Ans: $(-2\sqrt{2}, 7\sqrt{2})$
