EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$. [1 mark]

2. Find the rank of $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 4 & 6 \\ 1 & 4 & 2 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2 marks]

3. The matrix $\begin{bmatrix} 0 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1 mark]

4. Reduce $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 1 & 1 & 0 \\ -1 & 1 & 1 & 2 \end{bmatrix}$ to echelon form and find its rank. [2 marks]

5. What is the rank of the 5×3 null matrix? [1 mark]

6. Write the augmented matrix of the system $x + 3y + 3z = 4$, $5x - 4y - 2z = 1$ and $3y = -7$. [1 mark]

7. Examine whether the system $3x - 6y = -3$ and $-9x + 18y = 5$ is consistent. [2 marks]

8. Find k so that the system $7x - 3y = k$, $14x - 6y = 18$ has infinitely many solutions. [2 marks]

9. Show that the system $-6x + y = 0$, $x + y = 0$ has only the trivial solution. [2 marks]

10. Does the system $2x + 4y + 3z = 0$, $-2x + 5y - 3z = 0$ and $-2x + 4y - 3z = 0$ have a non-trivial solution? [3 marks]

11. Solve the homogeneous system $-5x - 3y + 4z = 0$, $4x + 3y - 2z = 0$ and $x + y = 0$. [5 marks]

12. Examine the consistency of $x + 4y - z = 6$, $-3x - 2y - 3z = 1$ and $x - 6y + 5z = -1$ by the rank method. [3 marks]

13. Examine the consistency of the system $x + 2y - z = 2$, $2x - 3z = 15$ and $-2y + 2z = -2$ by the rank method and solve it if consistent. [5 marks]

14. Show that the system $x + 2y = 7$, $x + y + 2z = 11$ and $4x + 6y + 4z = 36$ is consistent and has infinitely many solutions. Find them. [5 marks]

15. Using Cramer's rule, solve the system $2y + 2z = -8$, $x + 2y - z = -5$ and $-x + y - z = 3$. [5 marks]

16. Find the image of the point $(-2, -9)$ under reflection in the y-axis. [1 mark]

17. The point $(-1, -2)$ is reflected in the y-axis, and the image is then reflected in the x-axis. Find the final image. [2 marks]

18. Find the image of $(8, -1)$ under reflection in the line $y = \frac{4}{3}x$. [3 marks]

19. Find the image of $(-2, -3)$ when it is rotated anticlockwise about the origin through $180°$. [1 mark]

20. The point $(3, 2)$ is rotated anticlockwise about the origin through $135°$. Find its image. [3 marks]

Key sheet

Q	Answer
1	1
2	2
3	2
4	3
5	0
6	$\left[\begin{array}{ccc\|c} 1 & 3 & 3 & 4 \\ 5 & -4 & -2 & 1 \\ 0 & 3 & 0 & -7 \end{array}\right]$
7	Inconsistent: no solution
8	$k = 9$
9	Only $x = 0$, $y = 0$
10	Yes: $\lvert A\rvert = 0$, so it has non-trivial solutions
11	$(x, y, z) = (-2k, 2k, -k)$, k any number
12	Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A\lvert B] = 3$
13	$x = 3,\ y = -2,\ z = -3$
14	$(x, y, z) = (15 - 4k, 2k - 4, k)$, k any number
15	$x = -3,\ y = -2,\ z = -2$
16	$(2, -9)$
17	$(1, 2)$
18	$(-\frac{16}{5}, \frac{37}{5})$
19	$(2, 3)$
20	$(-\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

Solutions

1. Find the rank of $A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$. [1]

$\det A = (2)(-2) - (2)(-2) = 0$

A is singular but not null, and any non-zero element is a non-singular submatrix of order 1.

$\text{rank}(A) = 1$

Ans: 1

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2. Find the rank of $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 4 & 6 \\ 1 & 4 & 2 \end{bmatrix}$ by finding the largest non-singular square submatrix. [2]

$|A| = (8 - 24) + 0(4 - 6) + 4(8 - 4) = 0$

$|A| = 0$, so the rank is less than 3.

The submatrix $\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ has determinant $4 \neq 0$.

$\therefore \text{rank}(A) = 2$

Ans: 2

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3. The matrix $\begin{bmatrix} 0 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1]

The non-zero rows begin with 2, 3 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 2$

Ans: 2

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4. Reduce $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 1 & 1 & 0 \\ -1 & 1 & 1 & 2 \end{bmatrix}$ to echelon form and find its rank. [2]

Apply $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 + R_1$: $A \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -5 & 4 & 0 \\ 0 & -1 & 2 & 2 \end{bmatrix}$

Apply $R_2 \leftrightarrow R_3$: $\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & -5 & 4 & 0 \end{bmatrix}$

Apply $R_3 \to R_3 - 5R_2$: $\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & -6 & -10 \end{bmatrix}$

This is in echelon form with 3 non-zero rows.

$\therefore \text{rank}(A) = 3$

Ans: 3

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5. What is the rank of the 5×3 null matrix? [1]

The rank of a null matrix is defined to be 0.

Ans: 0

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6. Write the augmented matrix of the system $x + 3y + 3z = 4$, $5x - 4y - 2z = 1$ and $3y = -7$. [1]

Coefficient matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 5 & -4 & -2 \\ 0 & 3 & 0 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 4 \\ 5 & -4 & -2 & 1 \\ 0 & 3 & 0 & -7 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 4 \\ 5 & -4 & -2 & 1 \\ 0 & 3 & 0 & -7 \end{array}\right]$

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7. Examine whether the system $3x - 6y = -3$ and $-9x + 18y = 5$ is consistent. [2]

$\frac{a_1}{a_2} = \frac{3}{-9}$, $\frac{b_1}{b_2} = \frac{-6}{18}$, $\frac{c_1}{c_2} = \frac{-3}{5}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: the lines are parallel.

The system is inconsistent: no solution.

Ans: Inconsistent: no solution

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8. Find k so that the system $7x - 3y = k$, $14x - 6y = 18$ has infinitely many solutions. [2]

Infinitely many solutions need $\frac{7}{14} = \frac{-3}{-6} = \frac{k}{18}$

$\frac{k}{18} = \frac{1}{2} \Rightarrow k = 9$

Ans: $k = 9$

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9. Show that the system $-6x + y = 0$, $x + y = 0$ has only the trivial solution. [2]

$\frac{a_1}{a_2} = \frac{-6}{1}$, $\frac{b_1}{b_2} = \frac{1}{1}$

These are not equal (equivalently $|A| = -7 \neq 0$).

So the only solution is $x = y = 0$.

Ans: Only $x = 0$, $y = 0$

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10. Does the system $2x + 4y + 3z = 0$, $-2x + 5y - 3z = 0$ and $-2x + 4y - 3z = 0$ have a non-trivial solution? [3]

$|A| = 2(-15 + 12) - 4(6 - 6) + 3(-8 + 10) = 0$

$|A| = 0$, so the system has infinitely many solutions, including non-trivial ones.

Ans: Yes: $|A| = 0$, so it has non-trivial solutions

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11. Solve the homogeneous system $-5x - 3y + 4z = 0$, $4x + 3y - 2z = 0$ and $x + y = 0$. [5]

$|A| = -5(0 + 2) + 3(0 + 2) + 4(4 - 3) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = -k$ in the first two equations:

$-5x - 3y = 4k$, $4x + 3y = -2k$

Solving (determinant $-3 \neq 0$): $x = -2k,\ y = 2k$

These values also satisfy the third equation.

$(x, y, z) = (-2k, 2k, -k)$, k any number

Ans: $(x, y, z) = (-2k, 2k, -k)$, k **any number**

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12. Examine the consistency of $x + 4y - z = 6$, $-3x - 2y - 3z = 1$ and $x - 6y + 5z = -1$ by the rank method. [3]

Apply $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 10 & -6 & 19 \\ 0 & -10 & 6 & -7 \end{array}\right]$

Apply $R_3 \to R_3 + R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 10 & -6 & 19 \\ 0 & 0 & 0 & 12 \end{array}\right]$

$\text{rank}(A) = 2$ (non-zero rows of the coefficient part), $\text{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\text{rank}(A) = 2 \neq \text{rank}[A|B] = 3$

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13. Examine the consistency of the system $x + 2y - z = 2$, $2x - 3z = 15$ and $-2y + 2z = -2$ by the rank method and solve it if consistent. [5]

Apply $R_2 \to R_2 - 2R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -4 & -1 & 11 \\ 0 & -2 & 2 & -2 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -2 & 2 & -2 \\ 0 & -4 & -1 & 11 \end{array}\right]$

Apply $R_3 \to R_3 - 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & -5 & 15 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $-5z = 15$ gives $z = -3$

From row 2: $-2y + 2z = -2$, so $y = -2$

From row 1: $x + 2y - z = 2$, so $x = 3$

$x = 3,\ y = -2,\ z = -3$

Ans: $x = 3,\ y = -2,\ z = -3$

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14. Show that the system $x + 2y = 7$, $x + y + 2z = 11$ and $4x + 6y + 4z = 36$ is consistent and has infinitely many solutions. Find them. [5]

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & -1 & 2 & 4 \\ 0 & -2 & 4 & 8 \end{array}\right]$

Apply $R_3 \to R_3 - 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $-y + 2z = 4$, so $y = 2k - 4$

From row 1: $x + 2y = 7$, so $x = 15 - 4k$

$(x, y, z) = (15 - 4k, 2k - 4, k)$, k any number

Ans: $(x, y, z) = (15 - 4k, 2k - 4, k)$, k **any number**

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15. Using Cramer's rule, solve the system $2y + 2z = -8$, $x + 2y - z = -5$ and $-x + y - z = 3$. [5]

$$\Delta = \begin{vmatrix} 0 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = 0(-2+1) - 2(-1-1) + 2(1+2) = 10$$

$$\Delta_1 = \begin{vmatrix} -8 & 2 & 2 \\ -5 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix} = -8(-2+1) - 2(5+3) + 2(-5-6) = -30$$

$$\Delta_2 = \begin{vmatrix} 0 & -8 & 2 \\ 1 & -5 & -1 \\ -1 & 3 & -1 \end{vmatrix} = 0(5+3) + 8(-1-1) + 2(3-5) = -20$$

$$\Delta_3 = \begin{vmatrix} 0 & 2 & -8 \\ 1 & 2 & -5 \\ -1 & 1 & 3 \end{vmatrix} = 0(6+5) - 2(3-5) - 8(1+2) = -20$$

$x = \frac{\Delta_1}{\Delta} = \frac{-30}{10} = -3,\ y = \frac{\Delta_2}{\Delta} = \frac{-20}{10} = -2,\ z = \frac{\Delta_3}{\Delta} = \frac{-20}{10} = -2$

Ans: $x = -3,\ y = -2,\ z = -2$

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16. Find the image of the point $(-2, -9)$ under reflection in the y-axis. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$$

Ans: $(2, -9)$

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17. The point $(-1, -2)$ is reflected in the y-axis, and the image is then reflected in the x-axis. Find the final image. [2]

Reflection in the y-axis: $(-1, -2) \mapsto (1, -2)$

Reflection in the x-axis: $(1, -2) \mapsto (1, 2)$

In matrix form: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Ans: $(1, 2)$

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18. Find the image of $(8, -1)$ under reflection in the line $y = \frac{4}{3}x$. [3]

$\tan\theta = m = \frac{4}{3}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = -\frac{7}{25}$, $\sin 2\theta = \frac{2m}{1+m^2} = \frac{24}{25}$

Reflection matrix: $\begin{bmatrix} -\frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix}$

$$\begin{bmatrix} -\frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix} \begin{bmatrix} 8 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{16}{5} \\ \frac{37}{5} \end{bmatrix}$$

Image: $(-\frac{16}{5}, \frac{37}{5})$

Ans: $(-\frac{16}{5}, \frac{37}{5})$

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19. Find the image of $(-2, -3)$ when it is rotated anticlockwise about the origin through $180°$. [1]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Ans: $(2, 3)$

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20. The point $(3, 2)$ is rotated anticlockwise about the origin through $135°$. Find its image. [3]

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{5\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Image: $(-\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

Ans: $(-\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
