EXPRESSWALA ACADEMY

Maths 1A: Linear Equations

Question Paper, Set 10

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the rank of $A = \begin{bmatrix} 6 & 6 \\ 1 & 3 \end{bmatrix}$. [1 mark]

2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -1 & -2 \\ -4 & -4 & 10 \end{bmatrix}$. [2 marks]

3. The matrix $\begin{bmatrix} 5 & 0 & -2 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1 mark]

4. Reduce $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & -3 \\ -1 & -6 & -10 \end{bmatrix}$ to echelon form and find its rank. [2 marks]

5. What is the rank of the 3×7 null matrix? [1 mark]

6. Write the augmented matrix of the system $-3x + 6y + z = 10$, $2x - y + 5z = -1$ and $2x = 1$. [1 mark]

7. Show that the system $x + 4y = 8$ and $2x + 6y = 12$ has a unique solution, and find it. [2 marks]

8. Find k so that the system $3x - 6y = k$, $12x - 24y = 8$ has infinitely many solutions. [2 marks]

9. Show that the system $-2x - 4y = 0$, $-x + 2y = 0$ has only the trivial solution. [2 marks]

10. Examine whether the homogeneous system $-x - y - 2z = 0$, $-x - 3z = 0$ and $-y + 4z = 0$ has only the trivial solution. [3 marks]

11. Show that the system $x - 3y - 2z = 0$, $-x - z = 0$ and $-3x + 4y + z = 0$ has a non-trivial solution, and find all its solutions. [5 marks]

12. Examine the consistency of $x + 4y - 2z = 5$, $-8x - 2y + 8z = 5$ and $3x - 3y - 2z = 0$ by the rank method. [3 marks]

13. Examine the consistency of the system $x + 2y - 2z = 9$, $2x - y - 2z = 6$ and $x + y - z = 6$ by the rank method and solve it if consistent. [5 marks]

14. Show that the system $x + y = 2$, $-x - 2y + z = -5$ and $-2x - 3y + z = -7$ is consistent and has infinitely many solutions. Find them. [5 marks]

15. Solve $-2x - 2y + 3z = 17$, $-2x - 3y - z = 6$ and $-2x - 3y + z = 12$ by Cramer's rule. [5 marks]

16. Find the image of the point $(-7, 6)$ under reflection in the line $y = x$. [1 mark]

17. The point $(-5, 1)$ is reflected in the line $y = x$, and the image is then reflected in the line $y = -x$. Find the final image. [2 marks]

18. Find the image of $(4, 2)$ under reflection in the line $y = \frac{1}{2}x$. [3 marks]

19. Find the image of $(-1, -9)$ when it is rotated anticlockwise about the origin through $90°$. [1 mark]

20. The axes are rotated through $150°$ about the origin, the point staying fixed. Find the new coordinates of the point $(3, 5)$. [3 marks]

Key sheet

Q	Answer
1	2
2	2
3	2
4	2
5	0
6	$\left[\begin{array}{ccc\|c} -3 & 6 & 1 & 10 \\ 2 & -1 & 5 & -1 \\ 2 & 0 & 0 & 1 \end{array}\right]$
7	$x = 0,\ y = 2$
8	$k = 2$
9	Only $x = 0$, $y = 0$
10	No: only the trivial solution ($\lvert A\rvert = -3 \neq 0$)
11	$(x, y, z) = (k, k, -k)$, k any number
12	Inconsistent: $\operatorname{rank}(A) = 2 \neq \operatorname{rank}[A\lvert B] = 3$
13	$x = 3,\ y = 2,\ z = -1$
14	$(x, y, z) = (-k - 1, k + 3, k)$, k any number
15	$x = -3,\ y = -1,\ z = 3$
16	$(6, -7)$
17	$(5, -1)$
18	$(4, 2)$
19	$(9, -1)$
20	$(\frac{5}{2} - \frac{3\sqrt{3}}{2}, -\frac{5\sqrt{3}}{2} - \frac{3}{2})$

Solutions

1. Find the rank of $A = \begin{bmatrix} 6 & 6 \\ 1 & 3 \end{bmatrix}$. [1]

$\det A = (6)(3) - (6)(1) = 12$

$\det A \neq 0$, so A itself is a non-singular submatrix of order 2.

$\text{rank}(A) = 2$

Ans: 2

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2. Using determinants of square submatrices, find the rank of $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -1 & -2 \\ -4 & -4 & 10 \end{bmatrix}$. [2]

$|A| = (-10 - 8) + 3(30 - 8) + 3(-12 - 4) = 0$

$|A| = 0$, so the rank is less than 3.

The submatrix $\begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix}$ has determinant $8 \neq 0$.

$\therefore \text{rank}(A) = 2$

Ans: 2

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3. The matrix $\begin{bmatrix} 5 & 0 & -2 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form. Write its rank. [1]

The non-zero rows begin with 0, 1 zeros: the count increases row by row, and the zero rows are at the bottom.

So the matrix is in echelon form, and its rank is the number of non-zero rows.

$\text{rank} = 2$

Ans: 2

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4. Reduce $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & -3 \\ -1 & -6 & -10 \end{bmatrix}$ to echelon form and find its rank. [2]

Apply $R_3 \to R_3 + R_1$: $A \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix}$

Apply $R_3 \to R_3 - 2R_2$: $\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

This is in echelon form with 2 non-zero rows.

$\therefore \text{rank}(A) = 2$

Ans: 2

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5. What is the rank of the 3×7 null matrix? [1]

The rank of a null matrix is defined to be 0.

Ans: 0

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6. Write the augmented matrix of the system $-3x + 6y + z = 10$, $2x - y + 5z = -1$ and $2x = 1$. [1]

Coefficient matrix $A = \begin{bmatrix} -3 & 6 & 1 \\ 2 & -1 & 5 \\ 2 & 0 & 0 \end{bmatrix}$ (a missing variable has coefficient 0), constant matrix $B = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix}$.

The augmented matrix attaches B as an extra column: $[A \mid B] = \left[\begin{array}{ccc|c} -3 & 6 & 1 & 10 \\ 2 & -1 & 5 & -1 \\ 2 & 0 & 0 & 1 \end{array}\right]$

Ans: $\left[\begin{array}{ccc|c} -3 & 6 & 1 & 10 \\ 2 & -1 & 5 & -1 \\ 2 & 0 & 0 & 1 \end{array}\right]$

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7. Show that the system $x + 4y = 8$ and $2x + 6y = 12$ has a unique solution, and find it. [2]

$\frac{a_1}{a_2} = \frac{1}{2} \neq \frac{b_1}{b_2} = \frac{4}{6}$, so the lines meet in one point.

$X = A^{-1}B = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 12 \end{bmatrix}$

$x = 0,\ y = 2$

Ans: $x = 0,\ y = 2$

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8. Find k so that the system $3x - 6y = k$, $12x - 24y = 8$ has infinitely many solutions. [2]

Infinitely many solutions need $\frac{3}{12} = \frac{-6}{-24} = \frac{k}{8}$

$\frac{k}{8} = \frac{1}{4} \Rightarrow k = 2$

Ans: $k = 2$

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9. Show that the system $-2x - 4y = 0$, $-x + 2y = 0$ has only the trivial solution. [2]

$\frac{a_1}{a_2} = \frac{-2}{-1}$, $\frac{b_1}{b_2} = \frac{-4}{2}$

These are not equal (equivalently $|A| = -8 \neq 0$).

So the only solution is $x = y = 0$.

Ans: Only $x = 0, y = 0$

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10. Examine whether the homogeneous system $-x - y - 2z = 0$, $-x - 3z = 0$ and $-y + 4z = 0$ has only the trivial solution. [3]

$|A| = -(0 - 3) + (-4 - 0) - 2(1 - 0) = -3$

$|A| \neq 0$, so the system has only the trivial solution $x = y = z = 0$.

Ans: No: only the trivial solution ($|A| = -3 \neq 0$)

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11. Show that the system $x - 3y - 2z = 0$, $-x - z = 0$ and $-3x + 4y + z = 0$ has a non-trivial solution, and find all its solutions. [5]

$|A| = (0 + 4) + 3(-1 - 3) - 2(-4 - 0) = 0$

$|A| = 0$, so there are non-trivial solutions.

Put $z = -k$ in the first two equations:

$x - 3y = -2k$, $-x = -k$

Solving (determinant $-3 \neq 0$): $x = k,\ y = k$

These values also satisfy the third equation.

$(x, y, z) = (k, k, -k)$, k any number

Ans: $(x, y, z) = (k, k, -k)$, k **any number**

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12. Examine the consistency of $x + 4y - 2z = 5$, $-8x - 2y + 8z = 5$ and $3x - 3y - 2z = 0$ by the rank method. [3]

Apply $R_2 \to R_2 + 8R_1$, $R_3 \to R_3 - 3R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 5 \\ 0 & 30 & -8 & 45 \\ 0 & -15 & 4 & -15 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 5 \\ 0 & -15 & 4 & -15 \\ 0 & 30 & -8 & 45 \end{array}\right]$

Apply $R_3 \to R_3 + 2R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 5 \\ 0 & -15 & 4 & -15 \\ 0 & 0 & 0 & 15 \end{array}\right]$

$\mathrm{rank}(A) = 2$ (non-zero rows of the coefficient part), $\mathrm{rank}[A \mid B] = 3$.

The ranks differ, so the system is inconsistent: no solution.

Ans: Inconsistent: $\mathrm{rank}(A) = 2 \neq \mathrm{rank}[A|B] = 3$

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13. Examine the consistency of the system $x + 2y - 2z = 9$, $2x - y - 2z = 6$ and $x + y - z = 6$ by the rank method and solve it if consistent. [5]

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 9 \\ 0 & -5 & 2 & -12 \\ 0 & -1 & 1 & -3 \end{array}\right]$

Apply $R_2 \leftrightarrow R_3$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 9 \\ 0 & -1 & 1 & -3 \\ 0 & -5 & 2 & -12 \end{array}\right]$

Apply $R_3 \to R_3 - 5R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 9 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & -3 & 3 \end{array}\right]$

$\mathrm{rank}(A) = \mathrm{rank}[A \mid B] = 3$ = number of unknowns: consistent, unique solution.

From row 3: $-3z = 3$ gives $z = -1$

From row 2: $-y + z = -3$, so $y = 2$

From row 1: $x + 2y - 2z = 9$, so $x = 3$

$x = 3,\ y = 2,\ z = -1$

Ans: $x = 3,\ y = 2,\ z = -1$

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14. Show that the system $x + y = 2$, $-x - 2y + z = -5$ and $-2x - 3y + z = -7$ is consistent and has infinitely many solutions. Find them. [5]

Apply $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$: $[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & -1 & 1 & -3 \end{array}\right]$

Apply $R_3 \to R_3 - R_2$: $\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = \text{rank}[A \mid B] = 2 < 3$: consistent, infinitely many solutions.

Put $z = k$. From row 2: $-y + z = -3$, so $y = k + 3$

From row 1: $x + y = 2$, so $x = -k - 1$

$(x, y, z) = (-k - 1, k + 3, k)$, k any number

Ans: $(x, y, z) = (-k - 1, k + 3, k)$, k **any number**

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15. Solve $-2x - 2y + 3z = 17$, $-2x - 3y - z = 6$ and $-2x - 3y + z = 12$ by Cramer's rule. [5]

$$\Delta = \begin{vmatrix} -2 & -2 & 3 \\ -2 & -3 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -2(-3-3) + 2(-2-2) + 3(6-6) = 4$$

$$\Delta_1 = \begin{vmatrix} 17 & -2 & 3 \\ 6 & -3 & -1 \\ 12 & -3 & 1 \end{vmatrix} = 17(-3-3) + 2(6+12) + 3(-18+36) = -12$$

$$\Delta_2 = \begin{vmatrix} -2 & 17 & 3 \\ -2 & 6 & -1 \\ -2 & 12 & 1 \end{vmatrix} = -2(6+12) - 17(-2-2) + 3(-24+12) = -4$$

$$\Delta_3 = \begin{vmatrix} -2 & -2 & 17 \\ -2 & -3 & 6 \\ -2 & -3 & 12 \end{vmatrix} = -2(-36+18) + 2(-24+12) + 17(6-6) = 12$$

$x = \frac{\Delta_1}{\Delta} = \frac{-12}{4} = -3$, $y = \frac{\Delta_2}{\Delta} = \frac{-4}{4} = -1$, $z = \frac{\Delta_3}{\Delta} = \frac{12}{4} = 3$

Ans: $x = -3,\ y = -1,\ z = 3$

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16. Find the image of the point $(-7, 6)$ under reflection in the line $y = x$. [1]

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

Ans: $(6, -7)$

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17. The point $(-5, 1)$ is reflected in the line $y = x$, and the image is then reflected in the line $y = -x$. Find the final image. [2]

Reflection in the line $y = x$: $(-5, 1) \mapsto (1, -5)$

Reflection in the line $y = -x$: $(1, -5) \mapsto (5, -1)$

In matrix form: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

Ans: $(5, -1)$

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18. Find the image of $(4, 2)$ under reflection in the line $y = \frac{1}{2}x$. [3]

$\tan\theta = m = \frac{1}{2}$, so $\cos 2\theta = \frac{1-m^2}{1+m^2} = \frac{3}{5}$, $\sin 2\theta = \frac{2m}{1+m^2} = \frac{4}{5}$

Reflection matrix: $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Image: $(4, 2)$

Ans: $(4, 2)$

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19. Find the image of $(-1, -9)$ when it is rotated anticlockwise about the origin through $90°$. [1]

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -9 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

Ans: $(9, -1)$

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20. The axes are rotated through $150°$ about the origin, the point staying fixed. Find the new coordinates of the point $(3, 5)$. [3]

Turning the axes uses the transpose: $x' = x\cos\theta + y\sin\theta,\ y' = -x\sin\theta + y\cos\theta$

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \frac{3\sqrt{3}}{2} \\ -\frac{5\sqrt{3}}{2} - \frac{3}{2} \end{bmatrix}$$

New coordinates: $(\frac{5}{2} - \frac{3\sqrt{3}}{2}, -\frac{5\sqrt{3}}{2} - \frac{3}{2})$

Ans: $(\frac{5}{2} - \frac{3\sqrt{3}}{2}, -\frac{5\sqrt{3}}{2} - \frac{3}{2})$
