EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the domain of $\sin^{-1}(3x + 5)$. [1 mark]

2. Find the principal value of $\tan^{-1}\frac{1}{\sqrt{3}}$. [1 mark]

3. Evaluate $\cot^{-1}\frac{1}{\sqrt{3}} + \cos^{-1}\frac{1}{\sqrt{2}}$. [2 marks]

4. Find $\operatorname{cosec}^{-1}\left(\operatorname{cosec} -\frac{\pi}{9}\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\tan^{-1}\left(-\sqrt{3}\right) + \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$. [2 marks]

6. Express $\operatorname{cosec}^{-1}\frac{11}{2}$ as an inverse sine. [1 mark]

7. If $\sin^{-1} x = \frac{3\pi}{13}$, find $\cos^{-1} x$. [2 marks]

8. Prove that $\cos^{-1}\frac{12}{13} - \cos^{-1}\frac{3}{5} = -\cos^{-1}\frac{56}{65}$. [5 marks]

9. Find $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. [3 marks]

10. Solve $\tan^{-1} 8x + \tan^{-1} 12x = \frac{\pi}{4}$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{67}{69}$. [5 marks]

12. Express $\sin^{-1}\frac{9}{41}$ as an inverse cosine and as an inverse tangent. [2 marks]

13. Express $2\tan^{-1}\frac{1}{6}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Through which point on the y-axis does the graph of $y = \cot^{-1} x$ pass? [1 mark]

15. Find $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{5\pi}{6}\right)$. [2 marks]

16. Find the value of $\cos^{-1}\frac{1-x^2}{1+x^2}$ when $x = -\sqrt{3}$. [3 marks]

17. Using the telescoping sum, find $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21}$. [3 marks]

18. If $\sin^{-1}\frac{x}{5} + \sin^{-1}\frac{y}{3} = \theta$, prove that $9x^2 + 30xy\cos\theta + 25y^2 = 225\sin^2\theta$. [5 marks]

19. Which substitution removes the square root in $\sqrt{4 + x^2}$? What does the root become? [1 mark]

20. Simplify $\sin^{-1}\frac{x}{\sqrt{9+x^2}}$ for $x \in \mathbb{R}$. [2 marks]

Key sheet

Q	Answer
1	$[-2, -\frac{4}{3}]$
2	$\frac{\pi}{6}$
3	$\frac{7\pi}{12}$
4	$-\frac{\pi}{9}$
5	$-\frac{2\pi}{3}$
6	$\sin^{-1}\frac{2}{11}$
7	$\frac{7\pi}{26}$
8	Proved: $\cos^{-1}\frac{12}{13} - \cos^{-1}\frac{3}{5} = -\cos^{-1}\frac{56}{65}$
9	$\tan^{-1}\frac{1}{3}$
10	$x = \frac{1}{24}$
11	Proved
12	$\cos^{-1}\frac{40}{41}$, $\tan^{-1}\frac{9}{40}$
13	$\tan^{-1}\frac{12}{35}$, $\sin^{-1}\frac{12}{37}$, $\cos^{-1}\frac{35}{37}$
14	$\left(0, \frac{\pi}{2}\right)$
15	$\frac{\pi}{6}$
16	$\frac{2\pi}{3}$
17	$\tan^{-1}\frac{2}{3}$
18	Proved
19	$x = 2\tan\theta$; the root becomes $2\sec\theta$
20	$\tan^{-1}\frac{x}{3}$

Solutions

1. Find the domain of $\sin^{-1}(3x+5)$. [1]

$-1 \le 3x+5 \le 1$

$-6 \le 3x \le -4$

$-2 \le x \le -\frac{4}{3}$

Ans: $[-2, -\frac{4}{3}]$

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2. Find the principal value of $\tan^{-1}\frac{1}{\sqrt{3}}$. [1]

We need $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan\theta = \frac{1}{\sqrt{3}}$.

$\therefore \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Ans: $\frac{\pi}{6}$

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3. Evaluate $\cot^{-1}\frac{1}{\sqrt{3}} + \cos^{-1}\frac{1}{\sqrt{2}}$. [2]

$\cot^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{3},\ \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

Ans: $\frac{7\pi}{12}$

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4. Find $\operatorname{cosec}^{-1}\left(\operatorname{cosec}-\frac{\pi}{9}\right)$. [1]

$-\frac{\pi}{9}$ lies in the principal range $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

So $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ gives $-\frac{\pi}{9}$.

Ans: $-\frac{\pi}{9}$

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5. Using the rules for negative arguments, evaluate $\tan^{-1}\left(-\sqrt{3}\right) + \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$. [2]

$\tan^{-1}\left(-\sqrt{3}\right) = -\tan^{-1}\sqrt{3} = -\frac{\pi}{3}$

$\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = -\operatorname{cosec}^{-1}\frac{2}{\sqrt{3}} = -\frac{\pi}{3}$

$\text{Sum} = -\frac{2\pi}{3}$

Ans: $-\frac{2\pi}{3}$

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6. Express $\operatorname{cosec}^{-1}\frac{11}{2}$ as an inverse sine. [1]

$\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x},\ |x| \ge 1$

$\operatorname{cosec}^{-1}\frac{11}{2} = \sin^{-1}\frac{2}{11}$

Ans: $\sin^{-1}\frac{2}{11}$

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7. If $\sin^{-1}x = \frac{3\pi}{13}$, find $\cos^{-1}x$. [2]

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\cos^{-1}x = \frac{\pi}{2} - \frac{3\pi}{13} = \frac{7\pi}{26}$

Ans: $\frac{7\pi}{26}$

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8. Prove that $\cos^{-1}\frac{12}{13} - \cos^{-1}\frac{3}{5} = -\cos^{-1}\frac{56}{65}$. [5]

Let $x = \frac{12}{13}$, $y = \frac{3}{5}$.

$\sqrt{1-x^2} = \frac{5}{13},\ \sqrt{1-y^2} = \frac{4}{5}$

Here $x > y$.

$xy + \sqrt{1-x^2}\sqrt{1-y^2} = \frac{12}{13}\cdot\frac{3}{5} + \frac{5}{13}\cdot\frac{4}{5} = \frac{56}{65}$

So the formula gives $-\cos^{-1}\frac{56}{65}$.

Hence $\cos^{-1}\frac{12}{13} - \cos^{-1}\frac{3}{5} = -\cos^{-1}\frac{56}{65}$.

Ans: Proved: $\cos^{-1}\frac{12}{13} - \cos^{-1}\frac{3}{5} = -\cos^{-1}\frac{56}{65}$

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9. Find $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. [3]

Here $xy = \frac{1}{40} < 1$.

$\frac{x+y}{1-xy} = \frac{\frac{13}{40}}{\frac{39}{40}} = \frac{1}{3}$

$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{1}{3}$

Ans: $\tan^{-1}\frac{1}{3}$

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10. Solve $\tan^{-1} 8x + \tan^{-1} 12x = \frac{\pi}{4}$. [5]

$\tan^{-1}\frac{20x}{1-96x^2} = \frac{\pi}{4} \Rightarrow \frac{20x}{1-96x^2} = 1$

$96x^2 + 20x - 1 = 0$

$x = -\frac{1}{4}$ or $x = \frac{1}{24}$

For $x = -\frac{1}{4}$ both inverse tangents are negative, so the left side is negative and cannot equal $\frac{\pi}{4}$; reject it.

$\therefore x = \frac{1}{24}$

Ans: $x = \frac{1}{24}$

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11. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{67}{69}$. [5]

Let $x = \frac{1}{2}$, $y = \frac{1}{6}$, $z = \frac{1}{7}$, all positive.

$S_1 = x + y + z = \frac{17}{21},\ S_2 = xy + yz + zx = \frac{5}{28} < 1,\ S_3 = xyz = \frac{1}{84}$

$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\frac{S_1 - S_3}{1 - S_2} = \tan^{-1}\frac{\frac{67}{84}}{\frac{23}{28}}$

$= \tan^{-1}\frac{67}{69}$

Ans: Proved

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12. Express $\sin^{-1}\frac{9}{41}$ as an inverse cosine and as an inverse tangent. [2]

$x = \frac{9}{41} > 0$, $\sqrt{1-x^2} = \frac{40}{41}$.

$\sin^{-1}\frac{9}{41} = \cos^{-1}\sqrt{1-x^2} = \cos^{-1}\frac{40}{41}$

$\sin^{-1}\frac{9}{41} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \tan^{-1}\frac{9}{40}$

Ans: $\cos^{-1}\frac{40}{41}$, $\tan^{-1}\frac{9}{40}$

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13. Express $2\tan^{-1}\frac{1}{6}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$0 < x = \frac{1}{6} < 1$.

$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{\frac{1}{3}}{\frac{35}{36}} = \tan^{-1}\frac{12}{35}$

$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{\frac{1}{3}}{\frac{37}{36}} = \sin^{-1}\frac{12}{37}$

$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} = \cos^{-1}\frac{\frac{35}{36}}{\frac{37}{36}} = \cos^{-1}\frac{35}{37}$

Ans: $\tan^{-1}\frac{12}{35}$, $\sin^{-1}\frac{12}{37}$, $\cos^{-1}\frac{35}{37}$

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14. Through which point on the y-axis does the graph of $y = \cot^{-1}x$ pass? [1]

$\cot^{-1}0 = \frac{\pi}{2}$.

Ans: $\left(0, \frac{\pi}{2}\right)$

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15. Find $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{5\pi}{6}\right)$. [2]

$\frac{5\pi}{6}$ is not in the principal range $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

Using the periodic graph of $\operatorname{cosec}^{-1}(\operatorname{cosec}x)$, shift or reflect it into the range: the value is $\frac{\pi}{6}$.

Check: $\frac{\pi}{6}$ is in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ and $\operatorname{cosec}(\frac{\pi}{6}) = \operatorname{cosec}(\frac{5\pi}{6})$.

Ans: $\frac{\pi}{6}$

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16. Find the value of $\cos^{-1}\frac{1-x^2}{1+x^2}$ when $x = -\sqrt{3}$. [3]

$x = -\sqrt{3}$ lies in $x < 0$, where $\cos^{-1}\frac{1-x^2}{1+x^2} = -2\tan^{-1}x$.

$= \frac{2\pi}{3}$

Ans: $\frac{2\pi}{3}$

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17. Using the telescoping sum, find $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21}$. [3]

Each term is $\tan^{-1}\frac{1}{1+m(m+1)} = \tan^{-1}(m+1) - \tan^{-1}m$.

$\text{Sum} = \tan^{-1}5 - \tan^{-1}1$

$= \tan^{-1}\frac{5-1}{1+5\cdot1} = \tan^{-1}\frac{2}{3}$

Ans: $\tan^{-1}\frac{2}{3}$

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18. If $\sin^{-1}\frac{x}{5} + \sin^{-1}\frac{y}{3} = \theta$, prove that $9x^2 + 30xy\cos\theta + 25y^2 = 225\sin^2\theta$. [5]

Let $\sin^{-1}\frac{x}{5} = \alpha$, $\sin^{-1}\frac{y}{3} = \beta$, so $\alpha + \beta = \theta$.

Then $\sin\alpha = \frac{x}{5}$, $\cos\alpha = \sqrt{1 - \frac{x^2}{25}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \sqrt{1-\frac{x^2}{25}}\sqrt{1-\frac{y^2}{9}} - \frac{xy}{15}$

$\left(\cos\theta + \frac{xy}{15}\right)^2 = \left(1 - \frac{x^2}{25}\right)\left(1 - \frac{y^2}{9}\right)$

$\cos^2\theta + \frac{2xy}{15}\cos\theta + \frac{x^2y^2}{225} = 1 - \frac{x^2}{25} - \frac{y^2}{9} + \frac{x^2y^2}{225}$

$\frac{x^2}{25} + \frac{2xy}{15}\cos\theta + \frac{y^2}{9} = \sin^2\theta$

Multiplying by 225: $9x^2 + 30xy\cos\theta + 25y^2 = 225\sin^2\theta$.

Ans: Proved

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19. Which substitution removes the square root in $\sqrt{4+x^2}$? What does the root become? [1]

For $\sqrt{4+x^2}$ put $x = 2\tan\theta$.

Then the root becomes $2\sec\theta$.

Ans: $x = 2\tan\theta$**; the root becomes** $2\sec\theta$

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20. Simplify $\sin^{-1}\frac{x}{\sqrt{9+x^2}}$ for $x \in \mathbb{R}$. [2]

Put $x = 3\tan\theta$; then $\sqrt{9+x^2} = 3\sec\theta$.

$\sin^{-1}\frac{3\tan\theta}{3\sec\theta} = \sin^{-1}(\sin\theta) = \theta = \tan^{-1}\frac{x}{3}$

Ans: $\tan^{-1}\frac{x}{3}$
