EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 08

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For what values of x is $\cos^{-1}(4x+3)$ defined? [1 mark]

2. Find the principal value of $\cos^{-1}(-1)$. [1 mark]

3. Evaluate $\sin^{-1}\frac{\sqrt{3}}{2} - \sec^{-1}2$. [2 marks]

4. Find $\sin\left(\sin^{-1}\left(-\frac{3}{8}\right)\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\sec^{-1}(-1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. [2 marks]

6. Express $\tan^{-1}\left(-\frac{2}{5}\right)$ in terms of an inverse cotangent. [1 mark]

7. Solve $\cot^{-1}x = 2\tan^{-1}x$. [2 marks]

8. Prove that $\sin^{-1}\frac{24}{25} + \sin^{-1}\frac{4}{5} = \pi - \sin^{-1}\frac{4}{5}$. [5 marks]

9. Find $\tan^{-1}3 - \tan^{-1}(-2)$. [3 marks]

10. Solve $\tan^{-1}(x+3) + \tan^{-1}(x-3) = \tan^{-1}6$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{9} = \tan^{-1}\frac{49}{72}$. [5 marks]

12. Find $\tan\left(\cos^{-1}\frac{15}{17}\right)$. [2 marks]

13. Express $2\tan^{-1}\frac{3}{2}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Write the end points of the graph of $y = \cos^{-1}x$. [1 mark]

15. Find $\sec^{-1}\left(\sec\frac{4\pi}{3}\right)$. [2 marks]

16. Simplify $\tan^{-1}\frac{3x-x^3}{1-3x^2}$ for $x > \frac{1}{\sqrt{3}}$. [3 marks]

17. If $4\sin^{-1}x - 1\cos^{-1}x = 0$, find $4\sin^{-1}x + 1\cos^{-1}x$. [3 marks]

18. Prove that $\tan^{-1}\frac{1}{1+4\cdot5} + \tan^{-1}\frac{1}{1+5\cdot6} + \cdots + \tan^{-1}\frac{1}{1+(n+3)(n+4)} = \tan^{-1}\frac{n}{1+4(n+4)}$. [5 marks]

19. Find the greatest value of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$. [1 mark]

20. Simplify $\tan^{-1}\frac{\sqrt{36-x^2}}{x}$ for $0 < x < 6$. [2 marks]

Key sheet

Q	Answer
1	$[-1, -\frac{1}{2}]$
2	π
3	0
4	$-\frac{3}{8}$
5	$\frac{2\pi}{3}$
6	$-\pi + \cot^{-1}\left(-\frac{5}{2}\right)$
7	$x = \frac{1}{\sqrt{3}}$
8	Proved: $\sin^{-1}\frac{24}{25} + \sin^{-1}\frac{4}{5} = \pi - \sin^{-1}\frac{4}{5}$
9	$\frac{3\pi}{4}$
10	$x = 3$
11	Proved
12	$\frac{8}{15}$
13	$\pi - \tan^{-1}\frac{12}{5}$, $\pi - \sin^{-1}\frac{12}{13}$, $\cos^{-1}\left(-\frac{5}{13}\right)$
14	$(-1, \pi)$ and $(1, 0)$
15	$\frac{2\pi}{3}$
16	$-\pi + 3\tan^{-1}x$
17	$\frac{4\pi}{5}$
18	Proved
19	$\frac{5\pi^2}{4}$
20	$\cos^{-1}\frac{x}{6}$

Solutions

1. For what values of x is $\cos^{-1}(4x+3)$ defined? [1]

$-1 \le 4x+3 \le 1$

$-4 \le 4x \le -2$

$-1 \le x \le -\frac{1}{2}$

Ans: $[-1, -\frac{1}{2}]$

---

2. Find the principal value of $\cos^{-1}(-1)$. [1]

We need $\theta \in [0, \pi]$ with $\cos\theta = -1$.

$\therefore \cos^{-1}(-1) = \pi$

Ans: π

---

3. Evaluate $\sin^{-1}\frac{\sqrt{3}}{2} - \sec^{-1}2$. [2]

$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3},\ \sec^{-1}2 = \frac{\pi}{3}$

$\frac{\pi}{3} - \frac{\pi}{3} = 0$

Ans: 0

---

4. Find $\sin\left(\sin^{-1}\left(-\frac{3}{8}\right)\right)$. [1]

$\sin(\sin^{-1}x) = x$ for every x in the domain of $\sin^{-1}$.

$-\frac{3}{8}$ is in the domain, so the value is $-\frac{3}{8}$.

Ans: $-\frac{3}{8}$

---

5. Using the rules for negative arguments, evaluate $\sec^{-1}(-1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. [2]

$\sec^{-1}(-1) = \pi - \sec^{-1}1 = \pi - 0 = \pi$

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\frac{\sqrt{3}}{2} = -\frac{\pi}{3}$

$\text{Sum} = \frac{2\pi}{3}$

Ans: $\frac{2\pi}{3}$

---

6. Express $\tan^{-1}\left(-\frac{2}{5}\right)$ in terms of an inverse cotangent. [1]

$\tan^{-1}x = -\pi + \cot^{-1}\frac{1}{x},\ x < 0$

$\tan^{-1}\left(-\frac{2}{5}\right) = -\pi + \cot^{-1}\left(-\frac{5}{2}\right)$

Ans: $-\pi + \cot^{-1}\left(-\frac{5}{2}\right)$

---

7. Solve $\cot^{-1}x = 2\tan^{-1}x$. [2]

Let $\tan^{-1}x = t$; then $\cot^{-1}x = \frac{\pi}{2} - t$.

$\frac{\pi}{2} - t = 2t \Rightarrow t = \frac{\pi}{6}$

$\tan^{-1}x = \frac{\pi}{6} \Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Ans: $x = \frac{1}{\sqrt{3}}$

---

8. Prove that $\sin^{-1}\frac{24}{25}+\sin^{-1}\frac{4}{5}=\pi-\sin^{-1}\frac{4}{5}$. [5]

Let $x=\frac{24}{25}$, $y=\frac{4}{5}$.

$\sqrt{1-x^2}=\frac{7}{25},\ \sqrt{1-y^2}=\frac{3}{5}$

$x^2+y^2=\frac{976}{625}>1$ and both are positive.

$\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\pi-\sin^{-1}\left(\frac{24}{25}\cdot\frac{3}{5}+\frac{4}{5}\cdot\frac{7}{25}\right)=\pi-\sin^{-1}\frac{4}{5}$

Hence $\sin^{-1}\frac{24}{25}+\sin^{-1}\frac{4}{5}=\pi-\sin^{-1}\frac{4}{5}$.

Ans: Proved: $\sin^{-1}\frac{24}{25}+\sin^{-1}\frac{4}{5}=\pi-\sin^{-1}\frac{4}{5}$

---

9. Find $\tan^{-1}3-\tan^{-1}(-2)$. [3]

Here $x>0,\ y<0,\ xy<-1$.

$\frac{x-y}{1+xy}=\frac{5}{-5}=-1$

$\tan^{-1}3-\tan^{-1}(-2)=\pi+\tan^{-1}(-1)=\frac{3\pi}{4}$

Ans: $\frac{3\pi}{4}$

---

10. Solve $\tan^{-1}(x+3)+\tan^{-1}(x-3)=\tan^{-1}6$. [5]

$\tan^{-1}\frac{2x}{1-(x^2-9)}=\tan^{-1}6\Rightarrow\frac{2x}{10-x^2}=6$

$6x^2+2x-60=0$

$x=-\frac{10}{3}$ or $x=3$

For $x=-\frac{10}{3}$ both inverse tangents are negative (or the product condition fails), so the sum is not $\tan^{-1}6$; reject it.

$\therefore x=3$

Ans: $x=3$

---

11. Prove that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{9}=\tan^{-1}\frac{49}{72}$. [5]

Let $x=\frac{1}{3}$, $y=\frac{1}{6}$, $z=\frac{1}{9}$, all positive.

$S_1=x+y+z=\frac{11}{18},\ S_2=xy+yz+zx=\frac{1}{9}<1,\ S_3=xyz=\frac{1}{162}$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{S_1-S_3}{1-S_2}=\tan^{-1}\frac{\frac{49}{81}}{\frac{8}{9}}$

$=\tan^{-1}\frac{49}{72}$

Ans: Proved

---

12. Find $\tan\left(\cos^{-1}\frac{15}{17}\right)$. [2]

Let $\cos^{-1}\frac{15}{17}=\theta$, $\theta\in[0,\pi]$; then $\cos\theta=\frac{15}{17}$, $\sin\theta=\frac{8}{17}$.

$\tan\theta=\frac{\frac{8}{17}}{\frac{15}{17}}=\frac{8}{15}$

Ans: $\frac{8}{15}$

---

13. Express $2\tan^{-1}\frac{3}{2}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$x=\frac{3}{2}>1$.

$2\tan^{-1}x=\pi+\tan^{-1}\frac{2x}{1-x^2}=\pi+\tan^{-1}\left(-\frac{12}{5}\right)=\pi-\tan^{-1}\frac{12}{5}$

$2\tan^{-1}x=\pi-\sin^{-1}\frac{2x}{1+x^2}=\pi-\sin^{-1}\frac{12}{13}$

$2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}=\cos^{-1}\left(-\frac{5}{13}\right)$ (as $x\geq0$)

Ans: $\pi-\tan^{-1}\frac{12}{5}$, $\pi-\sin^{-1}\frac{12}{13}$, $\cos^{-1}\left(-\frac{5}{13}\right)$

---

14. Write the end points of the graph of $y = \cos^{-1} x$. [1]

$\cos^{-1}(-1) = \pi$ and $\cos^{-1} 1 = 0$.

Ans: $(-1, \pi)$ **and** $(1, 0)$

---

15. Find $\sec^{-1}\left(\sec \frac{4\pi}{3}\right)$. [2]

$\frac{4\pi}{3}$ is not in the principal range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Using the periodic graph of $\sec^{-1}(\sec x)$, shift or reflect it into the range: the value is $\frac{2\pi}{3}$.

Check: $\frac{2\pi}{3}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and $\sec(\frac{2\pi}{3}) = \sec(\frac{4\pi}{3})$.

Ans: $\frac{2\pi}{3}$

---

16. Simplify $\tan^{-1} \frac{3x - x^3}{1 - 3x^2}$ for $x > \frac{1}{\sqrt{3}}$. [3]

Put $x = \tan\theta$, so that the expression inside is a ratio of 3θ.

For $x > \frac{1}{\sqrt{3}}$ the angle has to be brought back into the principal range.

$\tan^{-1} \frac{3x - x^3}{1 - 3x^2} = -\pi + 3\tan^{-1} x$

Ans: $-\pi + 3\tan^{-1} x$

---

17. If $4\sin^{-1} x - 1\cos^{-1} x = 0$, find $4\sin^{-1} x + 1\cos^{-1} x$. [3]

Put $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$:

$4\sin^{-1} x - 1\left(\frac{\pi}{2} - \sin^{-1} x\right) = 0 \Rightarrow \sin^{-1} x = \frac{\pi}{10}$

$4\sin^{-1} x + 1\cos^{-1} x = 4 \cdot \frac{\pi}{10} + 1\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = \frac{4\pi}{5}$

Ans: $\frac{4\pi}{5}$

---

18. Prove that $\tan^{-1} \frac{1}{1+4\cdot5} + \tan^{-1} \frac{1}{1+5\cdot6} + \cdots + \tan^{-1} \frac{1}{1+(n+3)(n+4)} = \tan^{-1} \frac{n}{1+4(n+4)}$. [5]

The kth term is $\tan^{-1} \frac{1}{1+(k+3)(k+4)}$.

$\tan^{-1}(k+4) - \tan^{-1}(k+3) = \tan^{-1} \frac{(k+4)-(k+3)}{1+(k+4)(k+3)} = \tan^{-1} \frac{1}{1+(k+3)(k+4)}$

Adding for $k = 1, 2, \ldots, n$, all middle terms cancel:

$\text{Sum} = \tan^{-1}(n+4) - \tan^{-1} 4$

$= \tan^{-1} \frac{(n+4)-4}{1+4(n+4)} = \tan^{-1} \frac{n}{1+4(n+4)}$

Ans: Proved

---

19. Find the greatest value of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$. [1]

Put $\sin^{-1} x = t$, so $\cos^{-1} x = \frac{\pi}{2} - t$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The expression is a polynomial in t; its greatest value on this interval is $\frac{5\pi^2}{4}$.

Ans: $\frac{5\pi^2}{4}$

---

20. Simplify $\tan^{-1} \frac{\sqrt{36 - x^2}}{x}$ for $0 < x < 6$. [2]

Put $x = 6\cos\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$; then $\sqrt{36 - x^2} = 6\sin\theta$.

$\tan^{-1} \frac{6\sin\theta}{6\cos\theta} = \theta = \cos^{-1} \frac{x}{6}$

Ans: $\cos^{-1} \frac{x}{6}$
