EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the domain of $\sin^{-1}(2x + 7)$. [1 mark]

2. Find the principal value of $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [1 mark]

3. Evaluate $\operatorname{cosec}^{-1}\sqrt{2} + \cot^{-1} 1$. [2 marks]

4. Find $\cot^{-1}\left(\cot\frac{4\pi}{5}\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) + \operatorname{cosec}^{-1}(-1)$. [2 marks]

6. Express $\sec^{-1}\frac{10}{3}$ as an inverse cosine. [1 mark]

7. If $\operatorname{cosec}^{-1} x = \frac{2\pi}{7}$, find $\sec^{-1} x$. [2 marks]

8. Prove that $\cos^{-1}\frac{15}{17} + \cos^{-1}\frac{7}{25} = \cos^{-1}\left(-\frac{87}{425}\right)$. [5 marks]

9. Find $\tan^{-1}\frac{5}{6} + \tan^{-1}\frac{1}{11}$. [3 marks]

10. Solve $\tan^{-1} 6x + \tan^{-1} 9x = \frac{\pi}{4}$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} = \tan^{-1}\frac{73}{105}$. [5 marks]

12. Express $\tan^{-1}\frac{8}{15}$ as an inverse sine and as an inverse cosine. [2 marks]

13. Express $2\tan^{-1}\frac{2}{7}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Which value of y is never reached on the graph of $y = \operatorname{cosec}^{-1} x$? [1 mark]

15. Find $\cot^{-1}\left(\cot -\frac{\pi}{3}\right)$. [2 marks]

16. Find the value of $\sin^{-1}(3x - 4x^3)$ when $x = -\frac{1}{\sqrt{2}}$. [3 marks]

17. If $\sin^{-1}\frac{20}{x} + \sin^{-1}\frac{21}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3 marks]

18. If $\cos^{-1}\frac{x}{3} + \cos^{-1}\frac{y}{2} = \theta$, prove that $4x^2 - 12xy\cos\theta + 9y^2 = 36\sin^2\theta$. [5 marks]

19. Which substitution removes the square root in $\sqrt{36 - x^2}$? What does the root become? [1 mark]

20. Simplify $\tan^{-1}\frac{\sqrt{x^2-9}}{3}$ for $x \geq 3$. [2 marks]

Key sheet

Q	Answer
1	$[-4, -3]$
2	$-\frac{\pi}{4}$
3	$\frac{\pi}{2}$
4	$\frac{4\pi}{5}$
5	$\frac{\pi}{4}$
6	$\cos^{-1}\frac{3}{10}$
7	$\frac{3\pi}{14}$
8	Proved: $\cos^{-1}\frac{15}{17} + \cos^{-1}\frac{7}{25} = \cos^{-1}\left(-\frac{87}{425}\right)$
9	$\frac{\pi}{4}$
10	$x = \frac{1}{18}$
11	Proved
12	$\sin^{-1}\frac{8}{17}$, $\cos^{-1}\frac{15}{17}$
13	$\tan^{-1}\frac{28}{45}$, $\sin^{-1}\frac{28}{53}$, $\cos^{-1}\frac{45}{53}$
14	$y = 0$
15	$\frac{2\pi}{3}$
16	$-\frac{\pi}{4}$
17	$x = 29$
18	Proved
19	$x = 6\sin\theta$; the root becomes $6\cos\theta$
20	$\sec^{-1}\frac{x}{3}$

Solutions

1. Find the domain of $\sin^{-1}(2x+7)$. [1]

$-1 \le 2x+7 \le 1$

$-8 \le 2x \le -6$

$-4 \le x \le -3$

Ans: $[-4, -3]$

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2. Find the principal value of $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [1]

We need $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin\theta = -\frac{1}{\sqrt{2}}$.

$\therefore \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

Ans: $-\frac{\pi}{4}$

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3. Evaluate $\operatorname{cosec}^{-1}\sqrt{2} + \cot^{-1}1$. [2]

$\operatorname{cosec}^{-1}\sqrt{2} = \frac{\pi}{4},\ \cot^{-1}1 = \frac{\pi}{4}$

$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

Ans: $\frac{\pi}{2}$

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4. Find $\cot^{-1}\left(\cot\frac{4\pi}{5}\right)$. [1]

$\frac{4\pi}{5}$ lies in the principal range $(0, \pi)$.

So $\cot^{-1}(\cot\theta) = \theta$ gives $\frac{4\pi}{5}$.

Ans: $\frac{4\pi}{5}$

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5. Using the rules for negative arguments, evaluate $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) + \operatorname{cosec}^{-1}(-1)$. [2]

$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\frac{1}{\sqrt{2}} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\operatorname{cosec}^{-1}(-1) = -\operatorname{cosec}^{-1}1 = -\frac{\pi}{2}$

$\text{Sum} = \frac{\pi}{4}$

Ans: $\frac{\pi}{4}$

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6. Express $\sec^{-1}\frac{10}{3}$ as an inverse cosine. [1]

$\sec^{-1}x = \cos^{-1}\frac{1}{x},\ |x| \ge 1$

$\sec^{-1}\frac{10}{3} = \cos^{-1}\frac{3}{10}$

Ans: $\cos^{-1}\frac{3}{10}$

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7. If $\operatorname{cosec}^{-1}x = \frac{2\pi}{7}$, find $\sec^{-1}x$. [2]

$\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

$\sec^{-1}x = \frac{\pi}{2} - \frac{2\pi}{7} = \frac{3\pi}{14}$

Ans: $\frac{3\pi}{14}$

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8. Prove that $\cos^{-1}\frac{15}{17}+\cos^{-1}\frac{7}{25}=\cos^{-1}\left(-\frac{87}{425}\right)$. [5]

Let $x=\frac{15}{17}$, $y=\frac{7}{25}$.

$\sqrt{1-x^2}=\frac{8}{17},\ \sqrt{1-y^2}=\frac{24}{25}$

Here $x+y\geq 0$.

$xy-\sqrt{1-x^2}\sqrt{1-y^2}=\frac{15}{17}\cdot\frac{7}{25}-\frac{8}{17}\cdot\frac{24}{25}=-\frac{87}{425}$

So the formula gives $\cos^{-1}\left(-\frac{87}{425}\right)$.

Hence $\cos^{-1}\frac{15}{17}+\cos^{-1}\frac{7}{25}=\cos^{-1}\left(-\frac{87}{425}\right)$.

Ans: Proved: $\cos^{-1}\frac{15}{17}+\cos^{-1}\frac{7}{25}=\cos^{-1}\left(-\frac{87}{425}\right)$

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9. Find $\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}$. [3]

Here $xy=\frac{5}{66}<1$.

$\frac{x+y}{1-xy}=\frac{\frac{61}{66}}{\frac{61}{66}}=1$

$\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}=\tan^{-1}1=\frac{\pi}{4}$

Ans: $\frac{\pi}{4}$

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10. Solve $\tan^{-1}6x+\tan^{-1}9x=\frac{\pi}{4}$. [5]

$\tan^{-1}\frac{15x}{1-54x^2}=\frac{\pi}{4}\Rightarrow\frac{15x}{1-54x^2}=1$

$54x^2+15x-1=0$

$x=-\frac{1}{3}$ or $x=\frac{1}{18}$

For $x=-\frac{1}{3}$ both inverse tangents are negative, so the left side is negative and cannot equal $\frac{\pi}{4}$; reject it.

$\therefore x=\frac{1}{18}$

Ans: $x=\frac{1}{18}$

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11. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}=\tan^{-1}\frac{73}{105}$. [5]

Let $x=\frac{1}{4}$, $y=\frac{1}{5}$, $z=\frac{1}{6}$, all positive.

$S_1=x+y+z=\frac{37}{60},\ S_2=xy+yz+zx=\frac{1}{8}<1,\ S_3=xyz=\frac{1}{120}$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{S_1-S_3}{1-S_2}=\tan^{-1}\frac{\frac{73}{120}}{\frac{7}{8}}$

$=\tan^{-1}\frac{73}{105}$

Ans: Proved

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12. Express $\tan^{-1}\frac{8}{15}$ as an inverse sine and as an inverse cosine. [2]

$\sqrt{1+x^2}=\frac{17}{15}$

$\tan^{-1}\frac{8}{15}=\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\sin^{-1}\frac{8}{17}$

$\tan^{-1}\frac{8}{15}=\cos^{-1}\frac{1}{\sqrt{1+x^2}}=\cos^{-1}\frac{15}{17}$

Ans: $\sin^{-1}\frac{8}{17}$, $\cos^{-1}\frac{15}{17}$

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13. Express $2\tan^{-1}\frac{2}{7}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$0 < x = \frac{2}{7} < 1$.

$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{\frac{4}{7}}{\frac{45}{49}} = \tan^{-1}\frac{28}{45}$

$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{\frac{4}{7}}{\frac{53}{49}} = \sin^{-1}\frac{28}{53}$

$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} = \cos^{-1}\frac{\frac{45}{49}}{\frac{53}{49}} = \cos^{-1}\frac{45}{53}$

Ans: $\tan^{-1}\frac{28}{45}$, $\sin^{-1}\frac{28}{53}$, $\cos^{-1}\frac{45}{53}$

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14. Which value of y is never reached on the graph of $y = \operatorname{cosec}^{-1}x$? [1]

The range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Ans: $y = 0$

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15. Find $\cot^{-1}\left(\cot -\frac{\pi}{3}\right)$. [2]

$-\frac{\pi}{3}$ is not in the principal range $(0, \pi)$.

Using the periodic graph of $\cot^{-1}(\cot x)$, shift or reflect it into the range: the value is $\frac{2\pi}{3}$.

Check: $\frac{2\pi}{3}$ is in $(0, \pi)$ and $\cot(\frac{2\pi}{3}) = \cot(-\frac{\pi}{3})$.

Ans: $\frac{2\pi}{3}$

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16. Find the value of $\sin^{-1}(3x - 4x^3)$ when $x = -\frac{1}{\sqrt{2}}$. [3]

$x = -\frac{1}{\sqrt{2}}$ lies in $-1 \le x \le -\frac{1}{2}$, where $\sin^{-1}(3x - 4x^3) = -(\pi + 3\sin^{-1}x)$.

$= -\frac{\pi}{4}$

Ans: $-\frac{\pi}{4}$

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17. If $\sin^{-1}\frac{20}{x} + \sin^{-1}\frac{21}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3]

$\sin^{-1}\frac{20}{x} = \frac{\pi}{2} - \sin^{-1}\frac{21}{x} = \cos^{-1}\frac{21}{x}$.

So $\left(\frac{20}{x}\right)^2 + \left(\frac{21}{x}\right)^2 = 1$, giving $x = \sqrt{400 + 441} = 29$.

Ans: $x = 29$

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18. If $\cos^{-1}\frac{x}{3} + \cos^{-1}\frac{y}{2} = \theta$, prove that $4x^2 - 12xy\cos\theta + 9y^2 = 36\sin^2\theta$. [5]

Let $\cos^{-1}\frac{x}{3} = \alpha$, $\cos^{-1}\frac{y}{2} = \beta$, so $\alpha + \beta = \theta$.

Then $\cos\alpha = \frac{x}{3}$, $\sin\alpha = \sqrt{1 - \frac{x^2}{9}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{xy}{6} - \sqrt{1 - \frac{x^2}{9}}\sqrt{1 - \frac{y^2}{4}}$

$\left(\frac{xy}{6} - \cos\theta\right)^2 = \left(1 - \frac{x^2}{9}\right)\left(1 - \frac{y^2}{4}\right)$

$\frac{x^2y^2}{36} - \frac{2xy}{6}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{9} - \frac{y^2}{4} + \frac{x^2y^2}{36}$

$\frac{x^2}{9} - \frac{2xy}{6}\cos\theta + \frac{y^2}{4} = \sin^2\theta$

Multiplying by 36: $4x^2 - 12xy\cos\theta + 9y^2 = 36\sin^2\theta$.

Ans: Proved

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19. Which substitution removes the square root in $\sqrt{36 - x^2}$? What does the root become? [1]

For $\sqrt{36 - x^2}$ put $x = 6\sin\theta$.

Then the root becomes $6\cos\theta$.

Ans: $x = 6\sin\theta$**; the root becomes** $6\cos\theta$

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20. Simplify $\tan^{-1}\frac{\sqrt{x^2-9}}{3}$ for $x \geq 3$. [2]

Put $x = 3\sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right)$; then $\sqrt{x^2-9} = 3\tan\theta$.

$\tan^{-1}(\tan\theta) = \theta = \sec^{-1}\frac{x}{3}$

Ans: $\sec^{-1}\frac{x}{3}$