EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 06

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For what values of x is $\cos^{-1}(2x - 5)$ defined? [1 mark]

2. Find the principal value of $\operatorname{cosec}^{-1} \frac{2}{\sqrt{3}}$. [1 mark]

3. Evaluate $\cos^{-1} 0 + \tan^{-1} \frac{1}{\sqrt{3}}$. [2 marks]

4. Find $\tan\left(\tan^{-1} 9\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\sin^{-1}(-1) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$. [2 marks]

6. Express $\cot^{-1}\left(-\frac{7}{4}\right)$ in terms of an inverse tangent. [1 mark]

7. Solve $\tan^{-1} x = 2\cot^{-1} x$. [2 marks]

8. Prove that $\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{171}{221}$. [5 marks]

9. Find $\tan^{-1}(-2) + \tan^{-1}(-5)$. [3 marks]

10. Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} 12$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} = \tan^{-1}\frac{51}{47}$. [5 marks]

12. Express $\cos^{-1}\left(-\frac{3}{5}\right)$ as an inverse sine. [2 marks]

13. Express $2\tan^{-1} 4$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. The graph of $y = \sin^{-1} x$ is the mirror image of which curve in the line $y = x$? [1 mark]

15. Find $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$. [2 marks]

16. Simplify $\cos^{-1}(4x^3 - 3x)$ for $-\frac{1}{2} \le x \le \frac{1}{2}$. [3 marks]

17. If $\tan^{-1}\frac{4}{x} + \tan^{-1}\frac{9}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3 marks]

18. If $\sin^{-1}\frac{x}{2} + \sin^{-1}\frac{y}{5} = \theta$, prove that $25x^2 + 20xy\cos\theta + 4y^2 = 100\sin^2\theta$. [5 marks]

19. Find the least value of $(\cos^{-1} x)^2 - (\sin^{-1} x)^2$. [1 mark]

20. Simplify $\tan^{-1}\frac{x}{\sqrt{49-x^2}}$ for $|x| < 7$. [2 marks]

Key sheet

Q	Answer
1	$[2, 3]$
2	$\frac{\pi}{3}$
3	$\frac{2\pi}{3}$
4	9
5	$-\frac{2\pi}{3}$
6	$\pi - \tan^{-1}\frac{4}{7}$
7	$x = \sqrt{3}$
8	Proved: $\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{171}{221}$
9	$-\pi + \tan^{-1}\frac{7}{9}$
10	$x = \frac{4}{3}$
11	Proved
12	$\pi - \sin^{-1}\frac{4}{5}$
13	$\pi - \tan^{-1}\frac{8}{15}$, $\pi - \sin^{-1}\frac{8}{17}$, $\cos^{-1}\left(-\frac{15}{17}\right)$
14	$y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
15	$-\frac{\pi}{6}$
16	$2\pi - 3\cos^{-1} x$
17	$x = 6$
18	Proved
19	$-\frac{\pi^2}{4}$
20	$\sin^{-1}\frac{x}{7}$

Solutions

1. For what values of x is $\cos^{-1}(2x-5)$ defined? [1]

$-1 \le 2x-5 \le 1$

$4 \le 2x \le 6$

$2 \le x \le 3$

Ans: $[2,3]$

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2. Find the principal value of $\operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}$. [1]

We need $\theta \in \left[-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right]$ with $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$.

$\therefore \operatorname{cosec}^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{3}$

Ans: $\frac{\pi}{3}$

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3. Evaluate $\cos^{-1}0 + \tan^{-1}\frac{1}{\sqrt{3}}$. [2]

$\cos^{-1}0 = \frac{\pi}{2},\ \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

Ans: $\frac{2\pi}{3}$

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4. Find $\tan\left(\tan^{-1}9\right)$. [1]

$\tan(\tan^{-1}x) = x$ for every x in the domain of $\tan^{-1}$.

9 is in the domain, so the value is 9.

Ans: 9

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5. Using the rules for negative arguments, evaluate $\sin^{-1}(-1) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$. [2]

$\sin^{-1}(-1) = -\sin^{-1}1 = -\frac{\pi}{2}$

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

$\text{Sum} = -\frac{2\pi}{3}$

Ans: $-\frac{2\pi}{3}$

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6. Express $\cot^{-1}\left(-\frac{7}{4}\right)$ in terms of an inverse tangent. [1]

$\cot^{-1}x = \pi + \tan^{-1}\frac{1}{x},\ x<0$

$\cot^{-1}\left(-\frac{7}{4}\right) = \pi + \tan^{-1}\left(-\frac{4}{7}\right) = \pi - \tan^{-1}\frac{4}{7}$

Ans: $\pi - \tan^{-1}\frac{4}{7}$

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7. Solve $\tan^{-1}x = 2\cot^{-1}x$. [2]

Let $\cot^{-1}x = t$; then $\tan^{-1}x = \frac{\pi}{2} - t$.

$\frac{\pi}{2} - t = 2t \Rightarrow t = \frac{\pi}{6}$

$\cot^{-1}x = \frac{\pi}{6} \Rightarrow x = \cot\frac{\pi}{6} = \sqrt{3}$

Ans: $x = \sqrt{3}$

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8. Prove that $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{171}{221}$. [5]

Let $x=\frac{5}{13}$, $y=\frac{8}{17}$.

$\sqrt{1-x^2}=\frac{12}{13},\ \sqrt{1-y^2}=\frac{15}{17}$

$x^2+y^2=\frac{18041}{48841}\le 1$.

$\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\sin^{-1}\left(\frac{5}{13}\cdot\frac{15}{17}+\frac{8}{17}\cdot\frac{12}{13}\right)=\sin^{-1}\frac{171}{221}$

Hence $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{171}{221}$.

Ans: Proved: $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{171}{221}$

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9. Find $\tan^{-1}(-2)+\tan^{-1}(-5)$. [3]

Here $x,y<0,\ xy=10>1$.

$\frac{x+y}{1-xy}=\frac{-7}{-9}=\frac{7}{9}$

$\tan^{-1}(-2)+\tan^{-1}(-5)=-\pi+\tan^{-1}\frac{7}{9}$

Ans: $-\pi+\tan^{-1}\frac{7}{9}$

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10. Solve $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}12$. [5]

$\tan^{-1}\frac{2x}{1-(x^2-1)}=\tan^{-1}12\Rightarrow\frac{2x}{2-x^2}=12$

$12x^2+2x-24=0$

$x=-\frac{3}{2}$ or $x=\frac{4}{3}$

For $x=-\frac{3}{2}$ both inverse tangents are negative (or the product condition fails), so the sum is not $\tan^{-1}12$; reject it.

$\therefore x=\frac{4}{3}$

Ans: $x=\frac{4}{3}$

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11. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}=\tan^{-1}\frac{51}{47}$. [5]

Let $x=\frac{1}{2}$, $y=\frac{1}{5}$, $z=\frac{1}{6}$, all positive.

$S_1=x+y+z=\frac{13}{15},\ S_2=xy+yz+zx=\frac{13}{60}<1,\ S_3=xyz=\frac{1}{60}$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{S_1-S_3}{1-S_2}=\tan^{-1}\frac{\frac{17}{20}}{\frac{47}{60}}$

$=\tan^{-1}\frac{51}{47}$

Ans: Proved

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12. Express $\cos^{-1}\left(-\frac{3}{5}\right)$ as an inverse sine. [2]

$x=-\frac{3}{5}<0$, $\sqrt{1-x^2}=\frac{4}{5}$.

$\cos^{-1}x=\pi-\sin^{-1}\sqrt{1-x^2}$

$\cos^{-1}\left(-\frac{3}{5}\right)=\pi-\sin^{-1}\frac{4}{5}$

Ans: $\pi-\sin^{-1}\frac{4}{5}$

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13. Express $2\tan^{-1}4$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$x=4>1$.

$2\tan^{-1}x=\pi+\tan^{-1}\frac{2x}{1-x^2}=\pi+\tan^{-1}\left(-\frac{8}{15}\right)=\pi-\tan^{-1}\frac{8}{15}$

$2\tan^{-1}x=\pi-\sin^{-1}\frac{2x}{1+x^2}=\pi-\sin^{-1}\frac{8}{17}$

$2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}=\cos^{-1}\left(-\frac{15}{17}\right)$ (as $x\ge 0$)

Ans: $\pi-\tan^{-1}\frac{8}{15}$, $\pi-\sin^{-1}\frac{8}{17}$, $\cos^{-1}\left(-\frac{15}{17}\right)$

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14. The graph of $y = \sin^{-1} x$ is the mirror image of which curve in the line $y = x$? [1]

The graph of an inverse function is the reflection of the restricted function in $y = x$.

Ans: $y = \sin x$ **on** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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15. Find $\tan^{-1}\left(\tan \frac{5\pi}{6}\right)$. [2]

$\frac{5\pi}{6}$ is not in the principal range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Using the periodic graph of $\tan^{-1}(\tan x)$, shift or reflect it into the range: the value is $-\frac{\pi}{6}$.

Check: $-\frac{\pi}{6}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan(-\frac{\pi}{6}) = \tan(\frac{5\pi}{6})$.

Ans: $-\frac{\pi}{6}$

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16. Simplify $\cos^{-1}(4x^3 - 3x)$ for $-\frac{1}{2} \le x \le \frac{1}{2}$. [3]

Put $x = \cos\theta$, so that the expression inside is a ratio of 3θ.

For $-\frac{1}{2} \le x \le \frac{1}{2}$ the angle has to be brought back into the principal range.

$\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1} x$

Ans: $2\pi - 3\cos^{-1} x$

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17. If $\tan^{-1}\frac{4}{x} + \tan^{-1}\frac{9}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3]

Two inverse tangents add to $\frac{\pi}{2}$ when the product of the values is 1.

$\frac{4}{x} \cdot \frac{9}{x} = 1 \Rightarrow x^2 = 36 \Rightarrow x = 6$

Ans: $x = 6$

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18. If $\sin^{-1}\frac{x}{2} + \sin^{-1}\frac{y}{5} = \theta$, prove that $25x^2 + 20xy\cos\theta + 4y^2 = 100\sin^2\theta$. [5]

Let $\sin^{-1}\frac{x}{2} = \alpha$, $\sin^{-1}\frac{y}{5} = \beta$, so $\alpha + \beta = \theta$.

Then $\sin\alpha = \frac{x}{2}$, $\cos\alpha = \sqrt{1 - \frac{x^2}{4}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \sqrt{1 - \frac{x^2}{4}}\sqrt{1 - \frac{y^2}{25}} - \frac{xy}{10}$

$\left(\cos\theta + \frac{xy}{10}\right)^2 = \left(1 - \frac{x^2}{4}\right)\left(1 - \frac{y^2}{25}\right)$

$\cos^2\theta + \frac{2xy}{10}\cos\theta + \frac{x^2y^2}{100} = 1 - \frac{x^2}{4} - \frac{y^2}{25} + \frac{x^2y^2}{100}$

$\frac{x^2}{4} + \frac{2xy}{10}\cos\theta + \frac{y^2}{25} = \sin^2\theta$

Multiplying by 100: $25x^2 + 20xy\cos\theta + 4y^2 = 100\sin^2\theta$.

Ans: Proved

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19. Find the least value of $(\cos^{-1} x)^2 - (\sin^{-1} x)^2$. [1]

Put $\sin^{-1} x = t$, so $\cos^{-1} x = \frac{\pi}{2} - t$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The expression is a polynomial in t; its least value on this interval is $-\frac{\pi^2}{4}$.

Ans: $-\frac{\pi^2}{4}$

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20. Simplify $\tan^{-1}\frac{x}{\sqrt{49 - x^2}}$ for $|x| < 7$. [2]

Put $x = 7\sin\theta$; then $\sqrt{49 - x^2} = 7\cos\theta$.

$\tan^{-1}\frac{7\sin\theta}{7\cos\theta} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{7}$

Ans: $\sin^{-1}\frac{x}{7}$
