EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For what values of x is $\cos^{-1}(3x - 1)$ defined? [1 mark]

2. Find the principal value of $\sec^{-1}\frac{2}{\sqrt{3}}$. [1 mark]

3. Evaluate $\sec^{-1}\sqrt{2} + \operatorname{cosec}^{-1} 2$. [2 marks]

4. Find $\sec\left(\sec^{-1}\left(-\frac{5}{2}\right)\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \tan^{-1}\left(-\sqrt{3}\right)$. [2 marks]

6. Express $\tan^{-1}\left(-\frac{4}{3}\right)$ in terms of an inverse cotangent. [1 mark]

7. Solve $\cos^{-1} x = 2\sin^{-1} x$. [2 marks]

8. Prove that $\sin^{-1}\frac{8}{17} - \sin^{-1}\frac{3}{5} = \sin^{-1}\left(-\frac{13}{85}\right)$. [5 marks]

9. Find $\tan^{-1} 5 - \tan^{-1}\frac{2}{3}$. [3 marks]

10. Solve $\tan^{-1}(x + 2) + \tan^{-1}(x - 2) = \tan^{-1} 4$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{23}{24}$. [5 marks]

12. Find $\tan\left(\cos^{-1}\frac{7}{25}\right)$. [2 marks]

13. Express $2\tan^{-1}\frac{5}{2}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Which value of y is never reached on the graph of $y = \sec^{-1} x$? [1 mark]

15. Find $\sin^{-1}(\sin 6)$. [2 marks]

16. Simplify $\sin^{-1}\frac{2x}{1+x^2}$ for $x < -1$. [3 marks]

17. Using the telescoping sum, find $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21}$. [3 marks]

18. If $\cos^{-1}\frac{x}{4} + \cos^{-1}\frac{y}{5} = \theta$, prove that $25x^2 - 40xy\cos\theta + 16y^2 = 400\sin^2\theta$. [5 marks]

19. Find the greatest value of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$. [1 mark]

20. Simplify $\tan^{-1}\frac{\sqrt{16-x^2}}{x}$ for $0 < x < 4$. [2 marks]

Key sheet

Q	Answer
1	$[0, \frac{2}{3}]$
2	$\frac{\pi}{6}$
3	$\frac{5\pi}{12}$
4	$-\frac{5}{2}$
5	$\frac{\pi}{2}$
6	$-\pi + \cot^{-1}\left(-\frac{3}{4}\right)$
7	$x = \frac{1}{2}$
8	Proved: $\sin^{-1}\frac{8}{17} - \sin^{-1}\frac{3}{5} = \sin^{-1}\left(-\frac{13}{85}\right)$
9	$\frac{\pi}{4}$
10	$x = 2$
11	Proved
12	$\frac{24}{7}$
13	$\pi - \tan^{-1}\frac{20}{21}$, $\pi - \sin^{-1}\frac{20}{29}$, $\cos^{-1}\left(-\frac{21}{29}\right)$
14	$y = \frac{\pi}{2}$
15	$6 - 2\pi$
16	$-\pi - 2\tan^{-1}x$
17	$\tan^{-1}\frac{3}{11}$
18	Proved
19	$\frac{7\pi^3}{8}$
20	$\cos^{-1}\frac{x}{4}$

Solutions

1. For what values of x is $\cos^{-1}(3x-1)$ defined? [1]

$-1 \le 3x-1 \le 1$

$0 \le 3x \le 2$

$0 \le x \le \frac{2}{3}$

Ans: $[0, \frac{2}{3}]$

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2. Find the principal value of $\operatorname{sec}^{-1}\frac{2}{\sqrt{3}}$. [1]

We need $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ with $\sec\theta = \frac{2}{\sqrt{3}}$.

$\therefore \operatorname{sec}^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{6}$

Ans: $\frac{\pi}{6}$

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3. Evaluate $\operatorname{sec}^{-1}\sqrt{2} + \operatorname{cosec}^{-1} 2$. [2]

$\operatorname{sec}^{-1}\sqrt{2} = \frac{\pi}{4},\ \operatorname{cosec}^{-1} 2 = \frac{\pi}{6}$

$\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

Ans: $\frac{5\pi}{12}$

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4. Find $\sec\left(\operatorname{sec}^{-1}\left(-\frac{5}{2}\right)\right)$. [1]

$\sec(\operatorname{sec}^{-1} x) = x$ for every x in the domain of sec^{-1}.

$-\frac{5}{2}$ is in the domain, so the value is $-\frac{5}{2}$.

Ans: $-\frac{5}{2}$

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5. Using the rules for negative arguments, evaluate $\operatorname{sec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \tan^{-1}\left(-\sqrt{3}\right)$. [2]

$\operatorname{sec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \pi - \operatorname{sec}^{-1}\frac{2}{\sqrt{3}} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\tan^{-1}\left(-\sqrt{3}\right) = -\tan^{-1}\sqrt{3} = -\frac{\pi}{3}$

$\text{Sum} = \frac{\pi}{2}$

Ans: $\frac{\pi}{2}$

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6. Express $\tan^{-1}\left(-\frac{4}{3}\right)$ in terms of an inverse cotangent. [1]

$\tan^{-1} x = -\pi + \cot^{-1}\frac{1}{x},\ x < 0$

$\tan^{-1}\left(-\frac{4}{3}\right) = -\pi + \cot^{-1}\left(-\frac{3}{4}\right)$

Ans: $-\pi + \cot^{-1}\left(-\frac{3}{4}\right)$

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7. Solve $\cos^{-1} x = 2\sin^{-1} x$. [2]

Let $\sin^{-1} x = t$; then $\cos^{-1} x = \frac{\pi}{2} - t$.

$\frac{\pi}{2} - t = 2t \Rightarrow t = \frac{\pi}{6}$

$\sin^{-1} x = \frac{\pi}{6} \Rightarrow x = \sin\frac{\pi}{6} = \frac{1}{2}$

Ans: $x = \frac{1}{2}$

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8. Prove that $\sin^{-1}\frac{8}{17} - \sin^{-1}\frac{3}{5} = \sin^{-1}\left(-\frac{13}{85}\right)$. [5]

Let $x = \frac{8}{17}$, $y = \frac{3}{5}$.

$\sqrt{1-x^2} = \frac{15}{17},\ \sqrt{1-y^2} = \frac{4}{5}$

$x^2 + y^2 = \frac{4201}{7225} \le 1.$

$\sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) = \sin^{-1}\left(\frac{8}{17}\cdot\frac{4}{5} - \frac{3}{5}\cdot\frac{15}{17}\right) = \sin^{-1}\left(-\frac{13}{85}\right)$

Hence $\sin^{-1}\frac{8}{17} - \sin^{-1}\frac{3}{5} = \sin^{-1}\left(-\frac{13}{85}\right)$.

Ans: Proved: $\sin^{-1}\frac{8}{17} - \sin^{-1}\frac{3}{5} = \sin^{-1}\left(-\frac{13}{85}\right)$

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9. Find $\tan^{-1}5 - \tan^{-1}\frac{2}{3}$. [3]

Here $xy = \frac{10}{3} > -1$.

$\frac{x-y}{1+xy} = \frac{\frac{13}{3}}{\frac{13}{3}} = 1$

$\tan^{-1}5 - \tan^{-1}\frac{2}{3} = \tan^{-1}1 = \frac{\pi}{4}$

Ans: $\frac{\pi}{4}$

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10. Solve $\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}4$. [5]

$\tan^{-1}\frac{2x}{1-(x^2-4)} = \tan^{-1}4 \Rightarrow \frac{2x}{5-x^2} = 4$

$4x^2 + 2x - 20 = 0$

$x = -\frac{5}{2}$ or $x = 2$

For $x = -\frac{5}{2}$ both inverse tangents are negative (or the product condition fails), so the sum is not $\tan^{-1}4$; reject it.

$\therefore x = 2$

Ans: $x = 2$

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11. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{23}{24}$. [5]

Let $x = \frac{1}{3}$, $y = \frac{1}{4}$, $z = \frac{1}{5}$, all positive.

$S_1 = x+y+z = \frac{47}{60},\ S_2 = xy+yz+zx = \frac{1}{5} < 1,\ S_3 = xyz = \frac{1}{60}$

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{S_1 - S_3}{1-S_2} = \tan^{-1}\frac{\frac{23}{30}}{\frac{4}{5}}$

$= \tan^{-1}\frac{23}{24}$

Ans: Proved

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12. Find $\tan\left(\cos^{-1}\frac{7}{25}\right)$. [2]

Let $\cos^{-1}\frac{7}{25} = \theta$, $\theta \in [0, \pi]$; then $\cos\theta = \frac{7}{25}$, $\sin\theta = \frac{24}{25}$.

$\tan\theta = \frac{\frac{24}{25}}{\frac{7}{25}} = \frac{24}{7}$

Ans: $\frac{24}{7}$

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13. Express $2\tan^{-1}\frac{5}{2}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$x = \frac{5}{2} > 1.$

$2\tan^{-1}x = \pi + \tan^{-1}\frac{2x}{1-x^2} = \pi + \tan^{-1}\left(-\frac{20}{21}\right) = \pi - \tan^{-1}\frac{20}{21}$

$2\tan^{-1}x = \pi - \sin^{-1}\frac{2x}{1+x^2} = \pi - \sin^{-1}\frac{20}{29}$

$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} = \cos^{-1}\left(-\frac{21}{29}\right)$ (as $x \ge 0$)

Ans: $\pi - \tan^{-1}\frac{20}{21}$, $\pi - \sin^{-1}\frac{20}{29}$, $\cos^{-1}\left(-\frac{21}{29}\right)$

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14. Which value of y is never reached on the graph of $y = \sec^{-1} x$? [1]

The range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$; the line $y = \frac{\pi}{2}$ is never reached.

Ans: $y = \frac{\pi}{2}$

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15. Find $\sin^{-1}(\sin 6)$. [2]

6 is not in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Using the periodic graph of $\sin^{-1}(\sin x)$, shift or reflect it into the range: the value is $6 - 2\pi$.

Check: $6 - 2\pi$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(6 - 2\pi) = \sin(6)$.

Ans: $6 - 2\pi$

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16. Simplify $\sin^{-1} \frac{2x}{1+x^2}$ for $x < -1$. [3]

Put $x = \tan\theta$, so that the expression inside is a ratio of 2θ.

For $x < -1$ the angle has to be brought back into the principal range.

$\sin^{-1} \frac{2x}{1+x^2} = -\pi - 2\tan^{-1} x$

Ans: $-\pi - 2\tan^{-1} x$

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17. Using the telescoping sum, find $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{13} + \tan^{-1} \frac{1}{21}$. [3]

Each term is $\tan^{-1} \frac{1}{1+m(m+1)} = \tan^{-1}(m+1) - \tan^{-1} m$.

$\text{Sum} = \tan^{-1} 5 - \tan^{-1} 2$

$= \tan^{-1} \frac{5-2}{1+5\cdot 2} = \tan^{-1} \frac{3}{11}$

Ans: $\tan^{-1} \frac{3}{11}$

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18. If $\cos^{-1} \frac{x}{4} + \cos^{-1} \frac{y}{5} = \theta$, prove that $25x^2 - 40xy\cos\theta + 16y^2 = 400\sin^2\theta$. [5]

Let $\cos^{-1} \frac{x}{4} = \alpha$, $\cos^{-1} \frac{y}{5} = \beta$, so $\alpha + \beta = \theta$.

Then $\cos\alpha = \frac{x}{4}$, $\sin\alpha = \sqrt{1 - \frac{x^2}{16}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{xy}{20} - \sqrt{1 - \frac{x^2}{16}}\sqrt{1 - \frac{y^2}{25}}$

$\left(\frac{xy}{20} - \cos\theta\right)^2 = \left(1 - \frac{x^2}{16}\right)\left(1 - \frac{y^2}{25}\right)$

$\frac{x^2y^2}{400} - \frac{2xy}{20}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{16} - \frac{y^2}{25} + \frac{x^2y^2}{400}$

$\frac{x^2}{16} - \frac{2xy}{20}\cos\theta + \frac{y^2}{25} = \sin^2\theta$

Multiplying by 400: $25x^2 - 40xy\cos\theta + 16y^2 = 400\sin^2\theta$.

Ans: Proved

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19. Find the greatest value of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$. [1]

Put $\sin^{-1} x = t$, so $\cos^{-1} x = \frac{\pi}{2} - t$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The expression is a polynomial in t; its greatest value on this interval is $\frac{7\pi^3}{8}$.

Ans: $\frac{7\pi^3}{8}$

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20. Simplify $\tan^{-1} \frac{\sqrt{16-x^2}}{x}$ for $0 < x < 4$. [2]

Put $x = 4\cos\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$; then $\sqrt{16 - x^2} = 4\sin\theta$.

$\tan^{-1} \frac{4\sin\theta}{4\cos\theta} = \theta = \cos^{-1} \frac{x}{4}$

Ans: $\cos^{-1} \frac{x}{4}$
